REGULARITY OF C^1 SMOOTH SURFACES WITH PRESCRIBED p-MEAN CURVATURE IN THE HEISENBERG GROUP

JIH-HSIN CHENG, JENN-FANG HWANG, AND PAUL YANG

ABSTRACT. We consider a C^1 smooth surface with prescribed p(or H)-mean curvature in the 3-dimensional Heisenberg group. Assuming only the prescribed p-mean curvature $H \in C^0$, we show that any characteristic curve is C^2 smooth and its (line) curvature equals $-H$ in the nonsingular domain. By introducing characteristic coordinates and invoking the jump formulas along characteristic curves, we can prove that the Legendrian (or horizontal) normal gains one more derivative. Therefore the seed curves are C^2 smooth. We also obtain the uniqueness of characteristic and seed curves passing through a common point under some mild conditions, respectively. These results can be applied to more general situations.

1. Introduction and statement of the results

The p-minimal (or X-minimal, $\boldsymbol{H}$-minimal) surfaces have been studied extensively in the framework of geometric measure theory (e.g., [9], [8], [17]). Motivated by the isoperimetric problem in the Heisenberg group, one also studied nonzero constant p-mean curvature surfaces and the regularity problem (e.g., [16], [4], [13], [14], [20], [15], [18]). Starting from the work [6] (see also [5]), we studied the subject from the viewpoint of partial differential equations and that of differential geometry. In fact, the notion of p-mean curvature ("p-" stands for "pseudohermitian") can be defined for (hyper) surfaces in a pseudohermitian manifold. The Heisenberg group as a (flat) pseudohermitian manifold is the simplest model example, and represents a blow-up limit of general pseudohermitian manifolds. In this paper we will deal with the regularity problem, in particular, for a C^1 smooth surface with prescribed p-mean curvature in the 3-dimensional Heisenberg group $\boldsymbol{H^1}$. Since our results hold in quite general situations, we will just start with the general formulation.

Let Ω be a domain in R^m and $u : \Omega \to R$ be a $W^{1,1}$ function. Let $\vec{F}$ be an arbitrary (say, L^1) vector field on Ω, and $H \in L^\infty(\Omega)$. In [7] we consider the following energy functional:

$$\mathcal{F}(u) \equiv \int_\Omega |\nabla u + \vec{F}| + Hu$$

(we omit the Euclidean volume element). When $\vec{F} \equiv 0$, this is the energy functional of least gradient. When $m = 2$, $\vec{F} \equiv (-y, x)$, and $H = 0$, this is the p(or $\boldsymbol{H}$)-area

1991 *Mathematics Subject Classification.* Primary: 35L80; Secondary: 35J70, 32V20, 53A10, 49Q10.

Key words and phrases. Surface with prescribed p-mean curvature, Heisenberg group, characteristic curve, seed curve, characteristic coordinates.

of the graph defined by u in $\boldsymbol{H}^{\mathbf{1}}$. Let $\varphi \in W^{1,1}(\Omega)$ and $u_\varepsilon \equiv u + \varepsilon\varphi$ for $\varepsilon \in R$. We computed the first variation of u in the direction φ and obtained

$$\frac{d\mathcal{F}(u_\varepsilon)}{d\varepsilon} |_{\varepsilon=0\pm} = \pm \int_{S(u)} | \nabla\varphi | + \int_{\Omega \backslash S(u)} N^u \cdot \nabla\varphi + \int_\Omega H\varphi \tag{1.1}$$

(see (3.3) in [7]) where $S(u)$ denotes the singular set of u, consisting of the points (called singular points) where $\nabla u + \vec{F} = 0$, and $N^u \equiv N^u_{\vec{F}} \equiv \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$ (called Legendrian or horizontal normal). For a general $\vec{F}$ we cannot ignore the contribution of the first integral in the right-hand side of (1.1), caused by the singular set $S(u)$. For instance, we have $S(u) = \Omega$ in the case that $\vec{F} = 0$ and $u = 0$. For the study of the size of singular set, we refer the reader to [1] (in which singular set is called characteristic set). Let us denote the space of weakly differentiable functions by $W^1(\Omega)$ (see [10] for the precise definition). By (1.1) we define weak solutions as follows:

Definition 1.1. (Definition 3.2 in [7]) Let $\Omega \subset R^m$ be a bounded domain. Let $\vec{F}$ be an L^1_{loc} vector field and H be an L^1_{loc} function on Ω. We say $u \in W^1(\Omega)$ is a weak solution to equation $divN^u = H$ in Ω if for any $\varphi \in C^\infty_0(\Omega)$, there holds

$$\int_{S(u)} |\nabla\varphi| + \int_{\Omega\backslash S(u)} N^u \cdot \nabla\varphi + \int_\Omega H\varphi \geq 0. \tag{1.2}$$

In this situation, we also say that $divN^u = H$ in the weak sense.

With φ replaced by $-\varphi$ in (1.2), it follows that when the m-dimensional Hausdorff measure of $S(u)$, denoted as $\mathcal{H}^m(S(u))$, vanishes, then $u \in W^1(\Omega)$ is a weak solution to equation $divN^u = H$ in Ω if for any $\varphi \in C^\infty_0(\Omega)$, there holds

$$\int_\Omega N^u \cdot \nabla\varphi + \int_\Omega H\varphi = 0. \tag{1.3}$$

That is, if $\mathcal{H}^m(S(u)) = 0$, then $divN^u = H$ in the weak sense if (1.3) holds. In this paper, we consider only the situation that $S(u)$ is an empty set. A domain Ω is called nonsingular (for u) if $S(u)$ is empty. So we can use (1.3) as the definition of weak solutions for a nonsingular domain Ω. In this paper we assume u to be C^1, an assumption that is almost justified according to a structure theorem of Franchi, Serapioni, and Serra Cassano ([8]), modulo a measure zero set, the major part of a very general surface is $\boldsymbol{H}$-regular.

Theorem 1.1 ([8]) *If $E \subseteq \boldsymbol{H}^{\mathbf{1}}$ is a $\boldsymbol{H}$-Caccioppoli set, then $\partial^*_{\boldsymbol{H}}E$, the reduced boundary of E, is $\boldsymbol{H}$-rectifiable, that is*

$$\partial^*_{\boldsymbol{H}}E = \mathcal{N} \cup \bigcup_{h=1}^{\infty} K_h$$

where $\mathcal{H}^3_d(\mathcal{N}) = 0$ and K_h is a compact subset of a $\boldsymbol{H}$-regular surface.

On the other hand, it does not follow that a $\boldsymbol{H}$-regular surface can be represented by a C^1 graph. We thank F. Serra-Cassano and the referee for pointing out this fact.

Let $u \in C^1(\Omega)$, $\vec{F} \in C^0(\Omega)$, and Ω be nonsingular. Then N^u and $N^{u,\perp}$ exist and are in $C^0(\Omega)$, where $N^{u,\perp}$ denotes the rotation of N^u by $-\frac{\pi}{2}$ degrees. By

the O.D.E. theory ([11]), the integral curves of $N^{u,\perp}$ (N^u, resp.) exist and we call them characteristic (seed, resp.) curves. In [18], Pauls proved that for a C^1 smooth (weak) solution to equation $divN^u = H$ with $\vec{F} = (-y,\, x)$ and $H = 0$ in a nonsingular domain, the characteristic curves are straight lines and the seed curves are C^2 smooth under the condition that $N^u \in W^{1,1}$. We will show that the condition: $N^u \in W^{1,1}$ is not necessary while the curvature (along $N^{u,\perp}$ direction) of a characteristic curve is $-H$ under mild regularity condition on H (and nothing to do with precise form of $\vec{F}$; see Theorem A, Theorem D below)

For $u \in C^1(\Omega)$, $\vec{F} \in C^0(\Omega)$ in a nonsingular plane domain Ω, since $|N^u| = 1$, we can write $N^u = (\cos\theta,\, \sin\theta)$ locally with $\theta \in C^0$. We may forget u and consider $\theta \in C^0$ locally as an independent variable, then define $N \equiv (\cos\theta,\, \sin\theta)$ such that N and $N^\perp \equiv (\sin\theta,\, -\cos\theta)$ are C^0 vector fields. We also call the integral curves of $N^\perp$ (N, resp.) characteristic (seed, resp.) curves (see (1.6) below). Similarly to (1.3) for $\theta \in C^0(\Omega)$ we define

$$\text{div}\, N \equiv \text{div}(\cos\theta, \sin\theta) \equiv (\cos\theta)_x + (\sin\theta)_y = H \tag{1.4}$$

in the weak sense, meaning

$$\int_\Omega N \cdot \nabla\varphi + \int_\Omega H\varphi = 0.$$

for any $\varphi \in C_0^\infty(\Omega)$. On the other hand, equation $divN^u = H$ (i.e., θ arises from u) provides more information. For $u \in C^1$, $\vec{F} \in C^0$, let $D \equiv |\nabla u + \vec{F}|$. If $D \neq 0$ in Ω, write $N^{u,\perp} = (\sin\theta,\, -\cos\theta)$ and $\vec{F}^\perp = (F_2,\, -F_1)$ for $\vec{F} = (F_1,\, F_2)$. Then for any Lipschitzian domain $\Omega' \subset\subset \Omega$, we have

$$\int_{\Omega'} (DN^{u,\perp} - \vec{F}^\perp) \cdot \nabla\varphi = \int_{\Omega'} (u_y, -u_x) \cdot \nabla\varphi = 0 \tag{1.5}$$

(see the proof of Lemma 3.1) The "integrability" condition (1.5) (due to "$u_{yx} = u_{xy}$") makes equation $divN^u = H$ have more properties than (1.4).

Now we study (1.4) as a single equation for θ. For $\theta \in C^1$, (1.4) is a first-order equation whose characteristic curve $\Gamma = \{(x(\sigma),\, y(\sigma))\} \in C^1$ satisfies

$$\begin{aligned} \frac{dx}{d\sigma} &= \sin\theta(x(\sigma), y(\sigma)), \\ \frac{dy}{d\sigma} &= -\cos\theta(x(\sigma), y(\sigma)). \end{aligned} \tag{1.6}$$

Note that σ is a unit-speed parameter of Γ. For $\theta \in C^0$, we still use (1.6) as the definition of characteristic curves.

Theorem A. *Let Ω be a domain of R^2 and $H \in C^0(\Omega)$. Let $\theta \in C^0(\Omega)$ satisfy equation (1.4) in the weak sense, i.e.*

$$\int_\Omega (\cos\theta, \sin\theta) \cdot \nabla\varphi + \int_\Omega H\varphi = 0 \tag{1.7}$$

for all $\varphi \in C_0^\infty(\Omega)$ (cf. (1.3)). Let $\Gamma \subset \Omega$ be a (C^1 smooth) characteristic curve with σ being the unit-speed parameter, satisfying (1.6). Then Γ is C^2 smooth and the curvature of Γ (along $N^\perp$ direction) equals $-H$, that is, $\frac{d\theta}{d\sigma} = -H$.

Note that $\theta \in C^0$ implies $N \equiv (\cos\theta,\, \sin\theta) \in C^0$ and $N = N^u \equiv N^u_{\vec{F}} \in C^0$ if it arises from $u \in C^1$ and $\vec{F} \in C^0$. Recall that in Theorem A of [18], Pauls considered

the $H = 0$ case, in which Γ is a straight line under the condition that components of the horizontal Gauss map (i.e., $N^u_{\vec{F}}$ in our notation with $\vec{F} = (-y, x)$) are in $W^{1,1}(\Omega)$. In Theorem A above, if θ satisfies (1.7), we prove that Γ is a minimizer for the following energy functional:

$$L_H(\Gamma) \equiv |\Gamma| - \int_{\Omega_\Gamma} H dxdy$$

where $|\Gamma|$ denotes the length of Γ (see Section 2 for the definition of Ω_Γ). So the basic Calculus of Variations tells us that the curvature of Γ (along $N^\perp$ direction) equals $-H$ without invoking extra regularity assumption. Also H is only required to be C^0. In [15] Monti and Rickly considered the case of $H =$ constant for a convex isoperimetric set. We do not need convexity in Theorem A.

For $\theta \in C^0$, N ($N^\perp$, resp.) $\equiv (\cos\theta, \sin\theta)$ ($\equiv (\sin\theta, -\cos\theta)$, resp.) is a C^0 vector field. Then for any $p \in \Omega$, there exists at least one integral curve, i.e., seed curve (characteristic curve, resp.) passing through p. The uniqueness of integral curves for a C^0 vector field does not hold in general (see page 18 in [11]). In Section 3 we will prove uniqueness theorems B and B′ for characteristic and seed curves (see below). Let $p \in \Omega$ and $B_r(p) \equiv \{q \in \Omega \mid |q-p| < r\}$. Define $H_M(r) \equiv \max_{q \in \partial B_r(p)} |H(q)|$.

Theorem B. *(a) Let $\theta \in C^0(\Omega)$ and $H \in L^1_{loc}(\Omega)$ satisfy (1.7). Let $p \in \Omega$ and suppose there is $r_0 > 0$ such that $B_{r_0}(p) \subset\subset \Omega$ and*

$$\int_0^{r_0} H_M(r) dr < \infty. \tag{1.8}$$

Then there is r_1, $0 < r_1 \leq r_0$, such that there exists a unique seed curve passing through p in $B_{r_1}(p)$.

(b) Let $\theta \in C^0(\Omega)$ and $H \in C^{0,1}(\Omega)$ (Lipschitzian) satisfy (1.7). Then for any point $p \in \Omega$, we can find $r_1 > 0$ such that there exists a unique characteristic curve passing through p in $B_{r_1}(p)$.

In Theorem B (b), if H is only continuous, we can give an example for the nonuniqueness of characteristic curves (see Example 3.2). Note that u is not involved in Theorems A and B. Now we consider u. Let $u \in C^1$ and $\vec{F} = (F_1, F_2) \in C^1$. Recall that a point $p \in \Omega \subset R^2$ is called singular (nonsingular, resp.) if $\nabla u + \vec{F} = 0$ ($\neq 0$, resp.) at p. At a nonsingular point, we recall $N \equiv N^u \equiv \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$. We call Ω nonsingular if every point of Ω is not singular. We have another uniqueness theorem for characteristic curves.

Theorem B′. *Let $u : \Omega \subset R^2 \to R$ be a C^1 smooth function such that Ω is a nonsingular domain with $\vec{F} \in C^1(\Omega)$. Then for any point $p \in \Omega$, we can find $r_1 > 0$ such that $B_{r_1}(p) \subset \Omega$ and there exists a unique characteristic curve passing through p in $B_{r_1}(p)$.*

In Theorem B′ we only assume u, $\vec{F} \in C^1$, and do not use any property of H, in contrast to Theorem B (b). Even for the case $H = 0$, seed curves may only be C^1 smooth, but not C^2 smooth (see the remark after the proof of Theorem D in Section 5). However if $N \equiv (\cos\theta, \sin\theta)$ arises from u (i.e., $N = N^u$; see (1.3)),

Pauls ([18]) proved that when $u \in C^1$, $\theta \in C^0 \cap W^{1,1}$, and $H = 0$, then the seed curves are C^2 smooth. In Theorem D we prove the same conclusion under the condition that $u \in C^1$ ($\theta \in C^0$ follows) and $H \in C^1$ (in fact, that $H \in C^0$ and only C^1 in the N direction is enough).

If $u \in C^1(\Omega)$ also satisfies (1.3) with $\vec{F} \in C^1(\Omega)$ and $S(u)$ is empty in Ω, can we have higher order regularity for u, say, $u \in C^2$? This is impossible as shown by the following example. Let $u_g \equiv xy + g(y)$ where $g \in C^1 \backslash C^2$. Then u_g satisfies (1.3) with $H = 0$, $\vec{F} = (-y, x)$ on any nonsingular domain Ω for u_g. On the other hand, the characteristic and seed curves associated to u_g are all the same for different g's. That is, g determines the differentiability of u_g, but does not affect the shape of characteristic and seed curves. So we can prove that θ is in fact C^1 smooth (hence $N \in C^1$, but not $u \in C^2$) (see Theorem D below). Before doing this we need to introduce some kind of special coordinates.

Definition 1.2. Let N be a C^0 vector field with $|N| \equiv 1$ on a domain $\Omega \subset R^2$. A system of C^1 smooth local coordinates s, t is called a system of characteristic coordinates if s and t have the property that $\nabla s \parallel N^{\perp}$ and $\nabla t \parallel N$, i.e., ∇s and ∇t are parallel to $N^{\perp}$ and N, resp.. It follows that $\{t = \text{constant}\}$ are characteristic curves while $\{s = \text{constant}\}$ are seed curves.

Let $\Gamma_{(x,y)}(\sigma)$ ($\Lambda_{(x,y)}(\tau)$, resp.) denote a characteristic (seed, resp.) curve passing through (x, y), parametrized by the arc length σ (τ, resp.) with $\frac{d\Gamma_{(x,y)}(\sigma)}{d\sigma} = N^{\perp}$ ($\frac{d\Lambda_{(x,y)}(\tau)}{d\tau} = N$, resp.). For continuous functions g, f we write

$$(N^{\perp}g)(x, y) \equiv \frac{dg(\Gamma_{(x,y)}(\sigma))}{d\sigma} |_{\sigma=\sigma_0}$$

($(Nf)(x, y) \equiv \frac{df(\Lambda_{(x,y)}(\tau))}{d\tau} |_{\tau=\tau_0}$, resp.) if exists, where $(x,\ y) = \Gamma_{(x,y)}(\sigma_0)$ ($(x,\ y) = \Lambda_{(x,y)}(\tau_0)$, resp.). For a planar C^1 vector field $\vec{F} = (F_1,\ F_2)$, we define

$$rot\vec{F} := (F_2)_x - (F_1)_y.$$

We construct a system of characteristic coordinates in the following theorem.

Theorem C. *Let $u : \Omega \subset R^2 \to R$ be a C^1 smooth solution to (1.3) (Ω being nonsingular) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Then for any point $p_0 \in \Omega$ there exist a neighborhood $\Omega' \subset \Omega$ and real functions s, $t \in C^1(\Omega')$ such that $\{t =$ constants$\}$ and $\{s =$ constants$\}$ are characteristic curves and seed curves, respectively. Moreover, there are positive functions f, $g \in C^0(\Omega')$ such that*

$$\nabla s = fN^{\perp}, \nabla t = gDN. \tag{1.9}$$

Also Nf and $N^{\perp}g$ exist and are continuous in Ω'. In fact, f and g satisfy the following equations

$$Nf + fH = 0, N^{\perp}g + \frac{(rot\vec{F})g}{D} = 0. \tag{1.10}$$

For a perhaps smaller neighborhood $\Omega'' \subset \Omega'$ of p_0, the map $\Psi : (x,\ y) \in \Omega'' \to (s,\ t) \in \Psi(\Omega'')$ is a C^1 diffeomorphism such that

$$\Psi^*(\frac{ds^2}{f^2} + \frac{dt^2}{g^2D^2}) = dx^2 + dy^2. \tag{1.11}$$

We remark that the existence of C^1 smooth s can be proved for N satisfying (1.7) (i.e., not defined by u) instead of (1.3) (see Theorem 4.1).

Recall that for $u \in C^1(\Omega)$, $\vec{F} \in C^0(\Omega)$, and Ω being nonsingular, there exists $\theta \in C^0$ locally such that $N^u = (\cos\theta, \sin\theta)$.

Corollary C.1. *Suppose we are in the situation of Theorem C. Then θ is C^1 smooth in s and there holds*

$$\theta_s \equiv \frac{\partial\theta}{\partial s} = -\frac{H}{f}. \tag{1.12}$$

Note that by θ being C^1 smooth in s, we mean that $\theta_s \equiv \frac{\partial\theta}{\partial s}$ exists and is continuous. Since f is C^1 smooth in t, θ_s is also C^1 smooth in t if we assume that H has the same property according to (1.12). In fact, we can prove that θ is C^1 smooth in t too, and hence $\theta \in C^1$. That is, θ gains one derivative.

Theorem D. *Let $u : \Omega \subset R^2 \to R$ be a C^1 smooth (weak) solution to $divN^u = H$ in Ω (Ω being nonsingular) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Suppose $N(H)$ exists and is continuous. Then $\theta \in C^1$ and the characteristic and seed curves are C^2 smooth. Moreover, $N^{\perp}D$ exists and is continuous in Ω. In (s,t) coordinates near a given point as in Theorem C, we have*

$$\theta_t \equiv \frac{\partial\theta}{\partial t} = \frac{rot\vec{F}}{gD^2} - \frac{N^{\perp}(\log D)}{gD} = \frac{1}{gD^2}(rot\vec{F} - N^{\perp}D). \tag{1.13}$$

We remark that in case $H = 0$ or constant, we can prove Theorem D directly from the precise parametric expression of x or y. The situation $H =$ a nonzero constant arises from considering the boundary of a C^2 isoperimetric set. Pansu ([16]) conjectured that an isoperimetric set is congruent with a certain type of sphere. In [20], Ritoré and Rosales proved Pansu's conjecture for isoperimetric sets of class C^2 without any symmetry assumption. Later Monti and Rickly ([15]) obtained the same result for convex isoperimetric sets without regularity assumptions.

We outline the proof of $\theta \in C^1$ in Theorem D as follows. According to (1.12), we have good control for θ along the characteristic curves, i.e. the s-direction. If the control for θ fails along a seed curve, i.e. t-direction, say, at some s_0, then we show that it fails also for s near s_0. That is, the jump of a certain concerned quantity is kept in short "s-time" along the characteristic curves. This ends up to reach a contradiction. We borrow the idea of conveying information along the characteristic curves from the study of hyperbolic P.D.E. ([12]). After this paper was submitted, we were informed of recent results about regularity ([2], [3]) by the referee. In [3], Capogna, Citti, and Manfredini proved an interesting result, among others, that the Lipschitz minimizer obtained in Theorem A of [7] is actually $C^{1,\alpha}$ in a neighborhood of a nonsingular point under some extra condition. We would like to thank the referee for useful information, detailed comments, and grammatical suggestions.

2. **Curvature of characteristic curves-proof of Theorem A**

The following lemma should be a standard result. For completeness we give a proof.

Lemma 2.1. *Let $\theta \in C^0(\Omega)$ and $H \in L^1_{loc}(\Omega)$ satisfy (1.7). Then for any Lipschitzian domain $\Omega' \subset\subset \Omega$ and $\varphi \in C^1(\Omega)$, there holds*

$$\oint_{\partial\Omega'} \varphi N \cdot \nu = \int_{\Omega'} (\nabla\varphi) \cdot N + \varphi H \tag{2.1}$$

where $N \equiv (\cos\theta, \sin\theta)$ and ν denotes the unit outer normal to $\partial\Omega'$.

Proof. Take a Lipschitzian domain Ω'' such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Then there exists a sufficiently small number $\varepsilon_0 > 0$ such that for any $\psi \in C^1_0(\Omega'')$ and any ε, $0 < \varepsilon < \varepsilon_0$, the mollifier $\psi_\varepsilon \in C^\infty_0(\Omega)$. From (1.7) we have

$$0 = \int_\Omega N \cdot \nabla\psi_\varepsilon + H\psi_\varepsilon = \int_{\Omega''} N_\varepsilon \cdot \nabla\psi + H_\varepsilon\psi.$$

It then follows that *div* $N_\varepsilon = H_\varepsilon$ (strong sense) in Ω'' and since $\varphi \in C^1(\Omega)$, we have

$$\oint_{\partial\Omega'} \varphi N_\varepsilon \cdot \nu = \int_{\Omega'} (\nabla\varphi) \cdot N_\varepsilon + \varphi H_\varepsilon. \tag{2.2}$$

Since $N \in C^0(\Omega)$, N_ε converges to N uniformly in Ω'' while H_ε converges to H in $L^1(\Omega'')$ as $\varepsilon \to 0$. Therefore letting $\varepsilon \to 0$ in (2.2) gives (2.1).

□

By taking $\varphi \equiv 1$ in (2.1), we obtain

Corollary 2.2. *Let $\theta \in C^0(\Omega)$ and $H \in L^1_{loc}(\Omega)$ satisfy (1.7). Then for any Lipschitzian domain $\Omega' \subset\subset \Omega$, we have*

$$\oint_{\partial\Omega'} N \cdot \nu = \int_{\Omega'} H \tag{2.3}$$

where $N \equiv (\cos\theta, \sin\theta)$.

Proof. (of Theorem A). Without loss of generality, we may assume that locally Γ is a piece of a C^1 smooth graph $(x, y(x))$ with $0 \le x \le a$ $(a > 0)$ where $y(x) > 0$, $y'(x)$ is bounded, and the domain Ω_Γ surrounded by Γ and the three line segments connecting $(0, y(0))$, $(0, 0)$, $(a, 0)$, $(a, y(a))$ satisfies $\Omega_\Gamma \subset\subset \Omega$. We may also assume that $N^\perp \equiv (\sin\theta, -\cos\theta) = \frac{(1,y')}{\sqrt{1+(y')^2}}$ on Γ. Let Γ_ε be a family of small perturbations of Γ, having the same endpoints $(0, y(0))$, $(a, y(a))$ and described by $(x, y(x) + \varepsilon\varphi(x))$ $(\varphi \in C^\infty_0([0, a]))$ (see Figure 2.1)

The domain $\Omega_{\Gamma_\varepsilon}$ are defined similarly and $\Omega_{\Gamma_\varepsilon} \subset\subset \Omega$ for $|\varepsilon|$ small enough. Note that $y_\varepsilon(x) \equiv y(x) + \varepsilon\varphi(x) > 0$ (on $[0, a]$) also for $|\varepsilon|$ small enough. Let σ denote the arc length parameter. Let

$$G(\Gamma_\varepsilon) \equiv \int_{\Gamma_\varepsilon} N \cdot \nu d\sigma - \int_{\Omega_{\Gamma_\varepsilon}} H dxdy$$

where ν denotes the unit outer normal of $\Omega_{\Gamma_\varepsilon}$. Observe that

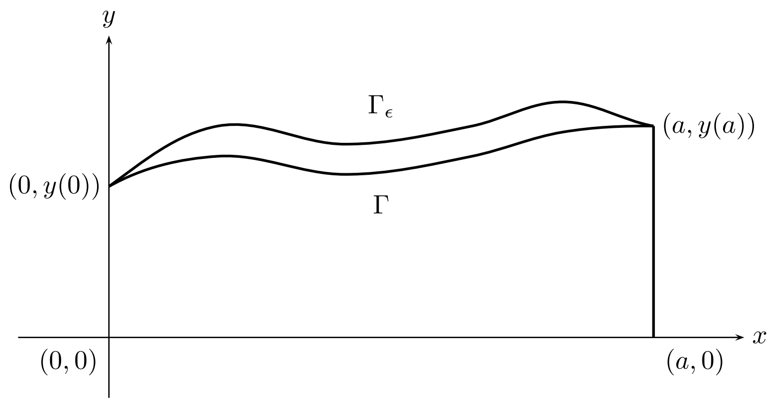

Figure 2.1

$$
\begin{aligned}
(2.4) \qquad & G(\Gamma_\varepsilon) - G(\Gamma) \\
= \ & \{\oint_{\partial\Omega_{\Gamma_\varepsilon}} N\cdot\nu d\sigma - \int_{\Omega_{\Gamma_\varepsilon}} Hdxdy\} - \{\oint_{\partial\Omega_\Gamma} N\cdot\nu d\sigma - \int_{\Omega_\Gamma} Hdxdy\} \\
= \ & 0 - 0 = 0
\end{aligned}
$$

by (2.3). Let $|\Gamma|$ denote the length of Γ. Along Γ we have $\nu = N$ since $N^\perp$ is tangent to Γ, a characteristic curve, $y(x) > 0$ and $N^\perp = \frac{(1,y')}{\sqrt{1+(y')^2}}$ on Γ. It follows that

$$
\begin{aligned}
(2.5) \quad |\Gamma| - \int_{\Omega_\Gamma} Hdxdy \quad &= \quad G(\Gamma) = G(\Gamma_\varepsilon) \text{ (by (2.4))} \\
&\leq \quad \int_{\Gamma_\varepsilon} |N\cdot\nu| d\sigma - \int_{\Omega_{\Gamma_\varepsilon}} Hdxdy \\
&\leq \quad |\Gamma_\varepsilon| - \int_{\Omega_{\Gamma_\varepsilon}} Hdxdy \ \text{ (by } |N\cdot\nu| \leq |N||\nu| = 1).
\end{aligned}
$$

Let

$$L_H(\Gamma) \equiv |\Gamma| - \int_{\Omega_\Gamma} Hdxdy.$$

We learn from (2.5) that $L_H(\Gamma)$ is the minimum of $L_H(\Gamma_\varepsilon)$. Therefore for $\varepsilon \in R\backslash\{0\}$, $|\varepsilon|$ small enough, we have

$$
\begin{aligned}
(2.6) \qquad 0 \quad &\leq \quad |\varepsilon|^{-1}\{(|\Gamma_\varepsilon| - \int_{\Omega_{\Gamma_\varepsilon}} Hdxdy) - (|\Gamma| - \int_{\Omega_\Gamma} Hdxdy)\} \\
&= \quad |\varepsilon|^{-1}(|\Gamma_\varepsilon| - |\Gamma|) - |\varepsilon|^{-1}(\int_{\Omega_{\Gamma_\varepsilon}} Hdxdy - \int_{\Omega_\Gamma} Hdxdy).
\end{aligned}
$$

Compute

$$
\begin{aligned}
(2.7)\qquad & \varepsilon^{-1}(|\Gamma_\varepsilon| - |\Gamma|) \\
= \;& \varepsilon^{-1}\{\int_0^a \sqrt{1+(y'(x)+\varepsilon\varphi'(x))^2}dx - \int_0^a \sqrt{1+(y'(x))^2}dx\} \\
= \;& \int_0^a \frac{2y'(x)\varphi'(x)+\varepsilon(\varphi'(x))^2}{\sqrt{1+(y'(x)+\varepsilon\varphi'(x))^2}+\sqrt{1+(y'(x))^2}}dx \\
\to \;& \int_0^a \frac{y'(x)}{\sqrt{1+(y'(x))^2}}\varphi'(x)dx
\end{aligned}
$$

while for some $\tilde{y}_\varepsilon(x)$ between $y(x)$ and $y(x)+\varepsilon\varphi(x)$ by the mean value theorem, we have

$$
\begin{aligned}
(2.8)\qquad & \varepsilon^{-1}(\int_{\Omega_{\Gamma_\varepsilon}} Hdxdy - \int_{\Omega_\Gamma} Hdxdy) \\
= \;& \varepsilon^{-1}\int_0^a H(x,\tilde{y}_\varepsilon(x))\varepsilon\varphi(x)dx \\
\to \;& \int_0^a H(x,y(x))\varphi(x)dx
\end{aligned}
$$

as $\varepsilon \to 0$ by Lebesgue's dominated convergence theorem.

From (2.6), (2.7), and (2.8) we obtain

$$
\frac{d}{dx}\left(\frac{y'(x)}{\sqrt{1+(y'(x))^2}}\right) = -H(x,y(x)) \tag{2.9}
$$

in the weak sense. Since $\frac{y'(x)}{\sqrt{1+(y'(x))^2}}$ and $H(x,y(x))$ are continuous in x, we actually have $\frac{y'(x)}{\sqrt{1+(y'(x))^2}} \in C^1$ with respect to x and (2.9) holds in the strong sense. Also it follows that $y \in C^2$ since $y'(x) = \frac{h(x)}{\sqrt{1-(h(x))^2}} \in C^1$ where $h(x) = \frac{y'(x)}{\sqrt{1+(y'(x))^2}}$. We have proved that Γ is C^2 smooth. Since $(\cos\theta, \sin\theta) = \frac{(1,y')}{\sqrt{1+(y')^2}}$, we have

$$
\frac{d\sin\theta}{dx} = -H
$$

by (2.9). It follows from $\frac{dx}{d\sigma} = \cos\theta$ that $-H = \cos\theta\frac{d\theta}{dx} = \frac{d\theta}{d\sigma}$.

□

3. Uniqueness of Characteristic and seed curves

Since N and $N^\perp$ are C^0 vector fields, there exist integral curves (called seed and characteristic curves, resp.) of N and $N^\perp$ passing through any given point. Uniqueness does not hold in general (see [11], page 18). However, if N satisfies (1.7), we have uniqueness.

Proof. (of Theorem B) First we will prove (a). Since $N \in C^0(\Omega)$, we can choose r_2, $0 < r_2 < r_0$, such that

$$
|N(q) - N(p)| \leq \frac{1}{2} \tag{3.1}
$$

for all $q \in B_{r_2}(p)$. Let Γ_1, Γ_2 are two seed curves passing through p. For $j = 1, 2$, let $\Gamma_j = \Gamma_j^+ \cup \Gamma_j^-$, $\Gamma_j^+ \cap \Gamma_j^- = \{p\}$ where Γ_j^+ (Γ_j^-, resp.) is the part of Γ_j emanating from p along the $+N$ ($-N$, resp.) direction. Then for any r, $0 \leq r < r_2$, there exist $p_{j,r}^{\pm}$, $j = 1, 2$ such that $\partial B_r(p) \cap \Gamma_j^{\pm} = \{p_{j,r}^{\pm}\}$ where $\partial B_0(p) \equiv \{p\}$. Suppose (a) fails to hold. Then without loss of generality, we may assume there exists r_4, $0 < r_4 < r_2$ such that $p_{1,r_4}^+ \neq p_{2,r_4}^+$ and there exists a unique r_3 depending on r_4 only such that $0 \leq r_3 < r_4$ and $p_{1,r_3}^+ = p_{2,r_3}^+$ (if $r_3 = 0$, $p_{1,r_3}^+ = p_{2,r_3}^+ = p$), $p_{1,r}^+ \neq p_{2,r}^+$ for $r_3 < r < r_4$ (see Figure 3.1).

Let ℓ_r denote the shorter arc of $\partial B_r(p)$ connecting $p_{1,r}^+$ and $p_{2,r}^+$. For perhaps smaller r_3, r_4 we have

$$N(p) \cdot \partial_r(q) \geq \frac{3}{4} \tag{3.2}$$

where $q \in \ell_r$ and $\partial_r(q) = \frac{q-p}{|q-p|}$ is the unit outer normal to ℓ_r for $r_3 < r < r_4$. It then follows from (3.1) and (3.2) that

$$\frac{1}{4} \leq N \cdot \partial_r \leq 1 \quad \text{on } \ell_r \tag{3.3}$$

for $r_3 < r < r_4$. Let Ω_r be the domain surrounded by $\Gamma_i^+ \cap (B_r(p) \backslash \bar{B}_{r_3}(p))$, $i = 1$, 2, and ℓ_r with vertices p_3 $(= p_{1,r_3}^+ = p_{2,r_3}^+)$, $p_{2,r}^+$, and $p_{1,r}^+$, where $\bar{B}_{r_3}(p) \equiv \{p\}$ if $r_3 = 0$. Let

$$h(r) \equiv \oint_{\partial\Omega_r} N \cdot \nu. \tag{3.4}$$

Observe that $N \cdot \nu = 0$ along Γ_1 and Γ_2 and $\nu = \partial_r$ on ℓ_r. It follows from (3.3) and (3.4) that

$$\frac{1}{4}|\ell_r| \leq h(r) \leq |\ell_r| \tag{3.5}$$

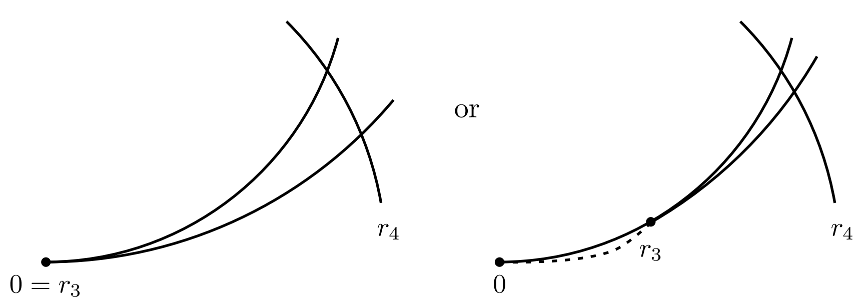

Figure 3.1

for $r_3 < r < r_4$ where $|\ell_r|$ denotes the arc length of ℓ_r. On the other hand, we compute from (2.3) that

$$h(r) \equiv \oint_{\partial\Omega_r} N \cdot \nu = \int_{\Omega_r} H$$

and hence by (3.5) we have

$$h'(r) = \int_{\ell_r} H \leq H_M(r)|\ell_r| \leq 4H_M(r)h(r) \tag{3.6}$$

for $r_3 < r < r_4$. We can therefore have

$$\begin{aligned} &\frac{d}{dr}[h(r)e^{-4\int_{r_3}^r H_M(r)dr}] \\ = \;& [h'(r) - 4H_M(r)h(r)]e^{-4\int_{r_3}^r H_M(r)dr} \leq 0 \end{aligned} \tag{3.7}$$

by (3.6). It follows from (3.7) that

$$h(r)e^{-4\int_{r_3}^r H_M(r)dr} \leq h(r_3) = 0. \tag{3.8}$$

By assumption (1.8) we have $\int_0^r H_M(r)dr < \infty$ and hence $h(r) = 0$ by (3.8) for $r_3 \leq r \leq r_4$. From (3.5) $|\ell_r| = 0$ for $r_3 \leq r \leq r_4$, which implies that $p^+_{1,r} = p^+_{2,r}$ for $r_3 < r < r_4$. We have reached a contradiction and hence proved (a).

Next we will prove (b). Suppose Γ_1 and Γ_2 are two characteristic curves passing through p. Without loss of generality we may assume that locally Γ_1 and Γ_2 are graphs $(x, y_1(x))$ and $(x, y_2(x))$, $|x| \leq x_1$ for some positive constant x_1, respectively and $p = (0, y_1(0)) = (0, y_2(0))$, $y_1'(0) = y_2'(0) = 0$. To prove (b) we need to show that $y_1(x) = y_2(x)$ on $[-x_0', x_0']$ for some small positive number $x_0' \leq x_1$. We will only show that $y_1(x) = y_2(x)$ on $[0, x_0]$ for some small positive number $x_0 \leq x_1$(see below) since a similar argument works for the interval of nonpositive numbers. We take $x_0 = \frac{1}{2H_1}$ for a large constant H_1 such that $|H| \leq H_1$ on Γ_1 and Γ_2 for $x \in [0, x_0]$.

By Theorem A the curvature of Γ_j, $j = 1, 2$, equals $-H$. Namely, we have (2.9) and hence (say, for $x \in [0, \frac{1}{2H_1}]$)

$$\frac{y_j'(x)}{\sqrt{1 + (y_j'(x))^2}} = -\int_0^x H(x, y_j(x))dx \tag{3.9}$$

for $j = 1, 2$. It follows that

$$\left|\frac{y_j'(x)}{\sqrt{1 + (y_j'(x))^2}}\right| \;\leq\; \frac{1}{2} \tag{3.10}$$

and hence

$$|y_j'(x)| \;\leq\; \frac{1}{\sqrt{3}} \tag{3.11}$$

for $x \in [0, \frac{1}{2H_1}]$, $j = 1, 2$. We want to prove that $y_2(x) = y_1(x)$ for all $x \in [0, \frac{1}{2H_1}]$. If not, we may assume

$$y_2(x) > y_1(x) \tag{3.12}$$

for all $x \in (0, \frac{1}{2H_1}]$ (otherwise we can find an interval $[a, b] \subset [0, \frac{1}{2H_1}]$, $a < b$, such that $y_1(a) = y_2(a)$, $y_1(b) = y_2(b)$, $y_1'(a) = y_2'(a)$, and $y_2(x) > y_1(x)$ (or

$y_2(x) < y_1(x)$) for $x \in (a, b)$. Applying a similar reasoning below to $[a, b]$ instead of $[0, \frac{1}{2H_1}]$, we will reach $y_2(x) = y_1(x)$ for all $x \in [a, b]$, a contradiction). From (3.9) we compute

$$\begin{aligned} (3.13) \qquad & \left|\frac{y_2'(x)}{\sqrt{1+(y_2'(x))^2}} - \frac{y_1'(x)}{\sqrt{1+(y_1'(x))^2}}\right| \\ \leq & \int_0^x |H(x, y_2(x)) - H(x, y_1(x))|dx \leq C_1 \int_0^x (y_2(x) - y_1(x))dx \end{aligned}$$

where C_1 is the Lipschitzian constant of H. Let

$$h(x) \equiv \int_0^x (y_2(x) - y_1(x))dx.$$

It follows that $h'(x) = y_2(x) - y_1(x)$ and $h''(x) = y_2'(x) - y_1'(x)$. On the other hand, observe that $f'(t) = (\frac{1}{1+t^2})^{3/2}$ for $f(t) = \frac{t}{(1+t^2)^{1/2}}$. By the mean-value theorem and (3.11), we have

$$\begin{aligned} (3.14) \qquad & \left|\frac{y_2'(x)}{\sqrt{1+(y_2'(x))^2}} - \frac{y_1'(x)}{\sqrt{1+(y_1'(x))^2}}\right| \\ \geq & (\frac{3}{4})^{3/2}|y_2'(x) - y_1'(x)|. \end{aligned}$$

From (3.14) and (3.13) we obtain the following differential inequality for h : ($C_2 = (\frac{4}{3})^{3/2}C_1$)

$$h''(x) \leq C_2 h(x). \tag{3.15}$$

Multiplying (3.15) by $h'(x)$ (> 0 by (3.12)) and integrating from 0 to $x \in (0, \frac{1}{2H_1}]$, we get

$$h'(x) \leq \sqrt{C_2} h(x) \tag{3.16}$$

in view of $h(x) > 0$, $h'(x) > 0$, and $h'(0) = h(0) = 0$. Writing (3.16) as $(\log h)'(x) \leq \sqrt{C_2}$ and then integrating from ε (> 0) to x, we obtain

$$h(x) \leq h(\varepsilon)e^{\sqrt{C_2}(x-\varepsilon)}. \tag{3.17}$$

Letting $\varepsilon \to 0$ in (3.17) gives $h(x) \equiv 0$ on $[0, \frac{1}{2H_1}]$, a contradiction. We have proved (b).

□

We remark that we can give an alternative proof of part (b) of Theorem B by applying Picard-Lindelöf's Theorem ([11], Theorem 1.1) to (3.9). Recall (see Section 1) that for $u \in C^1$ and $\vec{F} \in C^1$, let $D \equiv |\nabla u + \vec{F}|$, and if $D \neq 0$, we let $N \equiv N_{\vec{F}}^u \equiv \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$. Hence $N^\perp \equiv N_{\vec{F}}^{u,\perp} = D^{-1}$ $(u_y + F_2, -u_x - F_1)$ where we write $\vec{F} = (F_1, F_2)$. Recall the definition of $rot\vec{F}$ as follows:

$$rot\vec{F} = (F_2)_x - (F_1)_y.$$

It is easy to see that div $(DN^\perp) = rot\vec{F}$ if $u \in C^2$. Note that $rot\vec{F} = 2$ for $\vec{F} = (-y, x)$. For $u \in C^1$, we have the following result.

Lemma 3.1. *Let $u : \Omega \subset R^2 \to R$ be a C^1 smooth function such that $D \neq 0$ on Ω (i.e., Ω is a nonsingular domain) with $\vec{F} \in C^1(\Omega)$. Let $\Omega'' \subset\subset \Omega$ be a bounded Lipschitzian domain. Then for $\varphi \in C^1(\Omega)$ there holds*

$$\oint_{\partial\Omega''} \varphi DN^{\perp} \cdot \nu = \int_{\Omega''} \{(\nabla\varphi) \cdot (DN^{\perp}) + \varphi rot\vec{F}\}. \tag{3.18}$$

Proof. Write $DN^{\perp} = (u_y, -u_x) + (F_2, -F_1)$. Let v_ε denote a mollifier of v. Observe that $(u_y)_\varepsilon = (u_\varepsilon)_y$, $(u_x)_\varepsilon = (u_\varepsilon)_x$. It follows that

$$\begin{aligned} div((u_y)_\varepsilon, (-u_x)_\varepsilon) &= div((u_\varepsilon)_y, -(u_\varepsilon)_x) \\ &= (u_\varepsilon)_{yx} - (u_\varepsilon)_{xy} = 0. \end{aligned} \tag{3.19}$$

Now using the divergence theorem, we compute

$$\begin{aligned} &\oint_{\partial\Omega''} \varphi[((u_y)_\varepsilon, (-u_x)_\varepsilon) + (F_2, -F_1)] \cdot \nu \\ = &\int_{\Omega''} (\nabla\varphi) \cdot [((u_y)_\varepsilon, (-u_x)_\varepsilon) + (F_2, -F_1)] + \varphi div[((u_y)_\varepsilon, (-u_x)_\varepsilon) + (F_2, -F_1)] \\ = &\int_{\Omega''} (\nabla\varphi) \cdot [((u_y)_\varepsilon, (-u_x)_\varepsilon) + (F_2, -F_1)] + \varphi rot\vec{F} \end{aligned} \tag{3.20}$$

by (3.19) and noting that $div(F_2, -F_1) = rot\vec{F}$. Taking the limit $\varepsilon \to 0$ in (3.20) gives (3.18).

□

Proof. (of Theorem B′) Since $N^{\perp} \in C^0(\Omega)$, we can choose $r_2 > 0$ such that $B_{r_2}(p) \subset\subset \Omega$ and

$$|N^{\perp}(q) - N^{\perp}(p)| \leq \frac{1}{2} \tag{3.21}$$

for all $q \in B_{r_2}(p)$. Let Γ_1, Γ_2 be two characteristic curves passing through p. For $j = 1, 2$, let $\Gamma_j = \Gamma_j^+ \cup \Gamma_j^-$, $\Gamma_j^+ \cap \Gamma_j^- = \{p\}$ where Γ_j^+ (Γ_j^-, resp.) is the part of Γ_j emanating from p along the $+N^{\perp}$ ($-N^{\perp}$, resp.) direction.Then for any r, $0 \leq r < r_2$, there exist $p_{j,r}^{\pm}$, $j = 1, 2$ such that $\partial B_r(p) \cap \Gamma_j^{\pm} = \{p_{j,r}^{\pm}\}$. Suppose the conclusion is false. Then without loss of generality, we may assume there exists r_4, $0 < r_4 < r_2$ such that $p_{1,r_4}^+ \neq p_{2,r_4}^+$ and there exists a unique r_3 depending on r_4 only such that $0 \leq r_3 < r_4$ and $p_{1,r_3}^+ = p_{2,r_3}^+$ (if $r_3 = 0$, $p_{1,r_3}^+ = p_{2,r_3}^+ = p$), $p_{1,r}^+ \neq p_{2,r}^+$ for $r_3 < r < r_4$. Let ℓ_r denote the shortest arc of $\partial B_r(p)$ connecting $p_{1,r}^+$ and $p_{2,r}^+$. For perhaps smaller r_3, r_4 we have

$$N^{\perp}(p) \cdot \partial_r(q) \geq \frac{3}{4} \tag{3.22}$$

where $q \in \ell_r$ and $\partial_r(q) = \frac{q-p}{|q-p|}$ is the unit outer normal to ℓ_r for $r_3 < r < r_4$. It then follows from (3.21) and (3.22) that

$$\frac{1}{4} \leq N^{\perp} \cdot \partial_r \leq 1 \text{ on } \ell_r \tag{3.23}$$

for $r_3 < r < r_4$. Let Ω_r be the domain surrounded by $\Gamma_i^+ \cap (B_r(p) \backslash \bar{B}_{r_3}(p))$, $i = 1$, 2, and ℓ_r with vertices p_3 $(= p^+_{1,r_3} = p^+_{2,r_3})$, $p^+_{2,r}$, and $p^+_{1,r}$. Let

$$h(r) \equiv \oint_{\partial\Omega_r} DN^\perp \cdot \nu. \tag{3.24}$$

Observe that $N^\perp \cdot \nu = 0$ along Γ_1 and Γ_2 and $\nu = \partial_r$ on ℓ_r. It follows from (3.23) and (3.24) that

$$\frac{1}{4} C_1 |\ell_r| \leq h(r) \leq C_2 |\ell_r| \tag{3.25}$$

where $|\ell_r|$ denotes the arc length of ℓ_r, $C_1 \equiv \min_{\bar{B}_{r_4}(p)} D > 0$ since $D > 0$ on the nonsingular domain Ω, and $C_2 \equiv \max_{\bar{B}_{r_4}(p)} D$. On the other hand, we compute from (3.24) and (3.18) with $\varphi \equiv 1$ that

$$h(r) \equiv \oint_{\partial\Omega_r} DN^\perp \cdot \nu = \int_{\Omega_r} rot\vec{F}$$

and hence

$$h'(r) = \int_{\ell_r} rot\vec{F} \leq C_3 |\ell_r| \leq \frac{4C_3}{C_1} h(r) \tag{3.26}$$

by (3.25) for $r_3 < r < r_4$, where $|rot\vec{F}| \leq C_3 \equiv \max_{\bar{B}_{r_4}(p)} |rot\vec{F}|$. We can therefore have

$$\begin{aligned} &\frac{d}{dr}[h(r) e^{-\frac{4C_3 r}{C_1}}] \\ = \quad &[h'(r) - \frac{4C_3}{C_1} h(r)] e^{-\frac{4C_3 r}{C_1}} \leq 0 \end{aligned} \tag{3.27}$$

by (3.26). It follows from (3.27) that

$$h(r) e^{-\frac{4C_3 r}{C_1}} \leq h(r_3) e^{-\frac{4C_3 r_3}{C_1}} = 0.$$

Therefore $h(r) = 0$ for $r_3 < r < r_4$ and then from (3.25) we have $|\ell_r| = 0$ for $r_3 < r < r_4$ which implies that $p^+_{1,r} = p^+_{2,r}$ for $r_3 < r < r_4$, a contradiction.

□

Note that on the boundary of any Lipschitzian domain $\Omega' \subset\subset \Omega$ (Ω being nonsingular), $N^\perp \cdot \nu = N \cdot (\frac{dx}{d\sigma}, \frac{dy}{d\sigma})$ where σ is the unit-speed parameter and the unit tangent $(\frac{dx}{d\sigma}, \frac{dy}{d\sigma})$ is the rotation of the unit outer normal ν by $\frac{\pi}{2}$ degrees. It follows that for $\vec{F} = (-y, x)$, $DN^\perp \cdot \nu \, d\sigma = DN \cdot (dx, dy) = (u_x - y)\, dx + (u_y + x)\, dy = du + xdy - ydx$ which is the standard contact form of R^3, restricted to the surface $(x, y, u(x, y))$.

Example 3.2. We will define a family of curves to be the characteristic curves (for $N^\perp$ being its unit tangent vector field). Let $p_0 = (x_0, y_0)$ in the xy-plane. Case 1: For $y_0 - x_0^4 \geq 0$, we take $y = x^4 + (y_0 - x_0^4)$ to be the characteristic curve passing through p_0. Case 2: For $y_0 - x_0^4 < 0$ and $y_0 > 0$ (or equivalently, $0 < \frac{y_0}{x_0^4} < 1$ and $x_0 \neq 0$), we take $y = \frac{y_0}{x_0^4} x^4$ to be the characteristic curve passing through p_0. Case 3: For $y_0 \leq 0$, we take $y = y_0$ to be the characteristic curve passing through p_0.

Note that there are infinite number of the above-mentioned (characteristic) curves passing through the origin (0, 0) (see Figure 3.2 below).

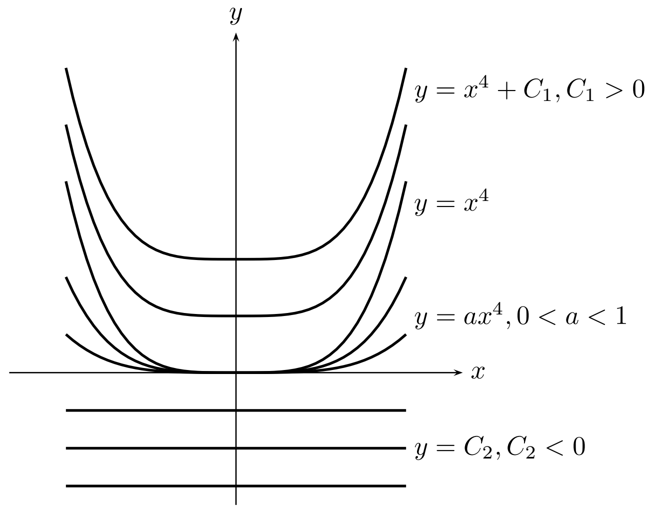

Figure 3.2

We compute the curvature of this family of curves as follows. For case 1, the curvature at p_0 equals $12x_0^2(1+16x_0^6)^{-3/2}$. For case 2, the curvature at p_0 equals $12(\frac{y_0}{x_0^4})x_0^2[1+16(\frac{y_0}{x_0^4})^2x_0^6]^{-3/2}$. For case 3, the curvature at p_0 vanishes. According to Theorem A the curvature of a characteristic curve is $-H$ (with this value (1.7) holds since $N = (\cos\theta, \sin\theta)$ is C^1 smooth away from (0, 0)). So we can easily verify that $H \in C^0$. On the other hand, we observe that

$$\frac{H(x,y)-H(x,0)}{y} = -\frac{12x^2(1+16x^6)^{-3/2}-0}{x^4} \to -\infty$$

as $x \to 0$ for $y = x^4$. Therefore $H \notin C^{0,1}$ in a neighborhood of (0, 0). Altogether this is a counterexample to Theorem B (b) if H is only continuous, but not Lipschitzian.

4. Characteristic coordinates

In this section we will introduce a system of characteristic coordinates (see Definition 1.2) for later use.

Theorem 4.1. *Let Ω be a domain of R^2 and $H \in C^0(\Omega)$. Let $\theta \in C^0(\Omega)$ satisfy (1.7), i.e.*

$$\int_\Omega (\cos\theta, \sin\theta)\cdot\nabla\varphi + \int_\Omega H\varphi = 0 \tag{4.1}$$

for all $\varphi \in C_0^\infty(\Omega)$. Then given a point $p_0 \in \Omega$, there exist a small neighborhood Ω' of p_0 and a function $s \in C^1(\Omega')$ such that $\nabla s = fN^\perp$ for some positive function $f \in C^0(\Omega')$ and the curves defined by $s = c$, constants, are seed curves. Moreover, Nf exists and is continuous. In fact, *f satisfies the following equation:*

$$Nf + fH = 0. \tag{4.2}$$

Proof. Without loss of generality we may assume $p_0 = (0,0)$, the origin, and $\theta(p_0) = \frac{\pi}{2}$. That is, $N \equiv (\cos\theta, \sin\theta) = (0, 1)$ at p_0. Let Υ denote the x-axis, i.e., $\Upsilon = \{ (x,y) \in \Omega \mid y = 0 \}$. Since θ is continuous, we can find a small ball $B_{r_1}(p_0) \subset\subset \Omega$ of radius $r_1 > 0$ such that for any point $q \in B_{r_1}(p_0)$, $|\theta(q) - \frac{\pi}{2}| << 1$ and there exists a seed curve passing through q and intersecting Υ at p. Now we define $s : B_{r_1}(p_0) \to R$ by

$$s(q) = s(p) = x \text{ if } p = (x,0). \tag{4.3}$$

Then s is well defined by the uniqueness of seed curves according to Theorem B (a). We can find smaller positive numbers $r_3 < r_2 < r_1$ such that $B_{r_3}(p_0) \subset\subset (-r_2, r_2) \times (-r_2, r_2) \subset\subset B_{r_1}(p_0)$ and for $c \in R$, if $\{s = c\} \cap B_{r_3}(p_0) \neq \phi$, then $\{s = c\} \cap (-r_2, r_2) \times (-r_2, r_2)$ is a graph, denoted by $(x^c(y), y)$, for $-r_2 < y < r_2$ (see Figure 4.1).

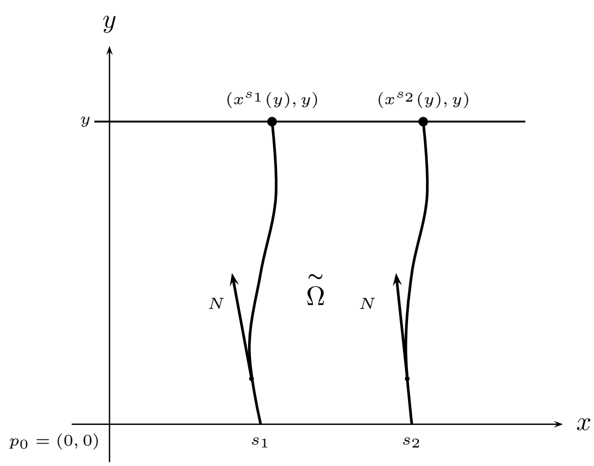

Figure 4.1

It follows that for $c_1 < c_2$

$$\begin{aligned} & s^{-1}((c_1, c_2)) \cap B_{r_3}(p_0) \\ = \ & [\bigcup_{c_1 < c < c_2} \{(x^c(y), y) | -r_2 < y < r_2\}] \cap B_{r_3}(p_0) \end{aligned}$$

is open. So $s \in C^0(B_{r_3}(p_0))$ and $x^{c_4}(y) > x^{c_3}(y)$ if and only if $c_4 > c_3$. Next we are going to prove $s \in C^1$. Given a point $(x_1, y) \in B_{r_3}(p_0)$, let $s_1 = s(x_1, y)$. Then $x_1 = x^{s_1}(y)$ by the definition of $x^c(y)$. Similarly for (x_2, y) near (x_1, y) and $x_2 > x_1$, let $s_2 = s(x_2, y)$. Then $x_2 = x^{s_2}(y)$ and $s_2 > s_1$. We want to compute the difference quotient

$$\frac{s(x_2, y) - s(x_1, y)}{x_2 - x_1} = \frac{s_2 - s_1}{x_2 - x_1} = \frac{s_2 - s_1}{x^{s_2}(y) - x^{s_1}(y)}. \tag{4.4}$$

Now let

$$A(y) \equiv \int_{x^{s_1}(y)}^{x^{s_2}(y)} \sin\theta(x, y) dx. \tag{4.5}$$

We want to know the relation between $A(y)$ and $A(0)$. Without loss of generality we may assume that $y > 0$. Let $\tilde{\Omega} \equiv \{\ (\varsigma,\ \eta) \mid 0 < \eta < y,\ x^{s_1}(\eta) < \varsigma < x^{s_2}(\eta)\}$. Recall that (2.3) (which is obtained from (4.1)) reads

$$\int_{\partial\tilde{\Omega}} N \cdot \nu = \int_{\tilde{\Omega}} H. \tag{4.6}$$

Observe that $\nu = \pm N^{\perp}$ along the seed curves $\{s = s_j\}$, $j = 1, 2$ while $\nu = (0, 1)$ on $\{\ (x, y) \mid x^{s_1}(y) < x < x^{s_2}(y)\}$ and $\nu = (0, -1)$ on $\{\ (x,\ 0) \mid s_1 < x < s_2\}$. It follows from (4.6) that (recall that $N = (\cos\theta, \sin\theta)$)

$$\int_{x^{s_1}(y)}^{x^{s_2}(y)} \sin\theta(x, y)dx + \int_{s_1}^{s_2} (-\sin\theta(x, 0))dx = \int_0^y (\int_{x^{s_1}(\eta)}^{x^{s_2}(\eta)} H(\varsigma, \eta)d\varsigma)d\eta. \tag{4.7}$$

Then by (4.7) we deduce that

$$A'(y) = \int_{x^{s_1}(y)}^{x^{s_2}(y)} H(x, y)dx. \tag{4.8}$$

By the mean value theorem, there exist $\varsigma_j = \varsigma_j(y,\ s_1,\ s_2)$, $j = 1, 2$ such that $x^{s_1}(y)$ $< \varsigma_j < x^{s_2}(y)$ and

$$\begin{aligned} A(y) &= (x^{s_2}(y) - x^{s_1}(y)) \sin\theta(\varsigma_1, y) \\ A'(y) &= (x^{s_2}(y) - x^{s_1}(y))H(\varsigma_2, y) \end{aligned} \tag{4.9}$$

in view of (4.5) and (4.8). By (4.9) we obtain

$$\frac{d \log A(y)}{dy} = \frac{A'(y)}{A(y)} = \frac{H(\varsigma_2, y)}{\sin\theta(\varsigma_1, y)} \tag{4.10}$$

(noting that $\sin\theta$ is close to 1 near p_0 where θ equals $\frac{\pi}{2}$ by assumption). Integrating both sides of (4.10) gives

$$\frac{A(y)}{A(0)} = \exp \int_0^y \frac{H(\varsigma_2, \eta)}{\sin\theta(\varsigma_1, \eta)} d\eta. \tag{4.11}$$

Observe that $A(0) = (s_2 - s_1)\sin\theta(\varsigma_1, 0)$ by (4.9). It then follows from (4.11) that

$$\frac{s_2 - s_1}{x^{s_2}(y) - x^{s_1}(y)} = \frac{\sin\theta(\varsigma_1(y), y)}{\sin\theta(\varsigma_1(0), 0)} \exp(-\int_0^y \frac{H(\varsigma_2(\eta), \eta)}{\sin\theta(\varsigma_1(\eta), \eta)} d\eta). \tag{4.12}$$

We have omitted the dependence of s_1 and s_2 for the expression of ς_1 and ς_2 in (4.12). Combining (4.4) and (4.12) and taking the limit $x_2 \to x_1$, we finally obtain

$$\frac{\partial s}{\partial x}(x_1, y) = \frac{\sin\theta(x_1, y)}{\sin\theta(s_1, 0)} \exp(-\int_0^y \frac{H(x^{s_1}(\eta), \eta)}{\sin\theta(x^{s_1}(\eta), \eta)} d\eta). \tag{4.13}$$

Here we have applied Lebesgue's Dominated Convergence Theorem since $\frac{H(\varsigma_2(\eta),\eta)}{\sin\theta(\varsigma_1(\eta),\eta)}$ is uniformly bounded and ς_j converges to x^{s_1} pointwise. Let τ denote the arc length (unit-speed) parameter of the seed curve $\{s = s_1\}$. By (4.3), the definition of s, we have $\frac{\partial s}{\partial \tau} = 0$. Note that $N d\tau = (dx,\ dy)$ along $\{s = s_1\} = \{\ (x^{s_1}(y),\ y)\ \}$. It then follows that

$$\begin{aligned} \frac{\partial s}{\partial y}(x_1, y) &= -\frac{\partial s}{\partial x}(x_1, y)\frac{dx^{s_1}(y)}{dy} = -\frac{\partial s}{\partial x}(x_1, y)\frac{\cos\theta(x_1, y)}{\sin\theta(x_1, y)} \\ &= f(x_1, y)(-\cos\theta(x_1, y)) \end{aligned} \tag{4.14}$$

by (4.13), where

$$f(x_1,y) \equiv \frac{1}{\sin\theta(s_1,0)}\exp\left(-\int_0^y \frac{H(x^{s_1}(\eta),\eta)}{\sin\theta(x^{s_1}(\eta),\eta)}d\eta\right). \tag{4.15}$$

Suppose $\tau = 0$ at $(s_1,\, 0)$ and $\tau = l$ at $(x_1,\, y)$. Recall that $Nd\tau = (dx,\, dy)$ and hence $\sin\theta(x^{s_1}(\eta),\, \eta)\, d\tau = d\eta$. So we can rewrite (4.15) as

$$f(x_1,y) = f(s_1,0)\exp\left(-\int_0^l H(\Lambda_{(s_1,0)}(\tau))d\tau\right) \tag{4.16}$$

where $\Lambda_{(s_1,0)}$ denotes the seed curve $\{s = s_1\}$ from $(s_1,\, 0)$ to $(x_1,\, y)$, parametrized by τ. Since $s \in C^0$, $\frac{H(x^{s_1}(\eta),\eta)}{\sin\theta(x^{s_1}(\eta),\eta)}$ is uniformly bounded, and $\frac{H(x^{s_2}(\eta),\eta)}{\sin\theta(x^{s_2}(\eta),\eta)}$ converges to $\frac{H(x^{s_1}(\eta),\eta)}{\sin\theta(x^{s_1}(\eta),\eta)}$ pointwise as $x_2 \to x_1$, we can apply Lebesgue's Dominated Convergence Theorem to conclude that f is continuous in x_1 in view of (4.15). On the other hand, f is continuous along the seed curve in view of (4.16). Together we have $f \in C^0$ near p_0. From (4.14), (4.13), and (4.15), we have proved $s \in C^1$ and

$$\nabla s = f(\sin\theta, -\cos\theta) = fN^{\perp}$$

with $f > 0$ and $f \in C^0$ near p_0. Moreover, recall that $(Nf)(x_1,y) \equiv df(\Lambda_{(s_1,0)}(\tau))/d\tau$ at $\tau = l$ (if exists). Now (4.2) easily follows from (4.16).

$\square$

Note that we do not need "u" (solution to $divN^u = H$, see (1.3)) to construct s in Theorem 4.1.

Theorem 4.2. *Let $u : \Omega \subset R^2 \to R$ be a C^1 smooth function and $\vec{F} \in C^1(\Omega)$ such that $D \equiv |\nabla u + \vec{F}| \neq 0$ on Ω (i.e., Ω is a nonsingular domain). Suppose we have uniqueness of characteristic curves passing through a common point. Then given a point $p_0 \in \Omega$, there exist a neighborhood Ω' of p_0 and a function $t \in C^1(\Omega')$ such that $\nabla t = gDN(u)$ for some positive $g \in C^0(\Omega')$ and the curves defined by t = c, constants, are characteristic curves. Moreover, $N^{\perp}g$ exists and is continuous. In fact, g satisfies the following equation:*

$$N^{\perp}g + \frac{(rot\vec{F})g}{D} = 0. \tag{4.17}$$

Proof. The idea similar to that in the proof of Theorem 4.1 works by switching the role of N and $N^{\perp}$. Let us sketch a proof. Without loss of generality we may assume $p_0 = (0,0)$, the origin, and $\theta(p_0) = \frac{\pi}{2}$. That is, $N^{\perp} \equiv (\sin\theta, -\cos\theta) = (1,0)$ at p_0. Let Υ denote the y-axis. Since θ is continuous, we can find a small ball $B_{r_1}(p_0)$ of radius $r_1 > 0$ such that for any point $q \in B_{r_1}(p_0)$ there exists a characteristic curve passing through q and intersecting with Υ at p. Now we define $t : B_{r_1}(p_0) \to R$ by

$$t(q) = t(p) = y \text{ if } p = (0,y). \tag{4.18}$$

Then t is well defined by the uniqueness of characteristic curves according to the assumption. We can find smaller positive numbers $r_3 < r_2 < r_1$ such that $B_{r_3}(p_0) \subset\subset (-r_2,r_2) \times (-r_2,r_2) \subset\subset B_{r_1}(p_0)$ and for $c \in R$, if $\{t = c\} \cap B_{r_3}(p_0) \neq \phi$, then $\{t = c\} \cap (-r_2,r_2) \times (-r_2,r_2)$ is a graph, denoted by $(x,\, y^c(x))$, for $-r_2 < x < r_2$. By a similar argument as in the proof of Theorem 4.1 we can prove that t is

C^0 in $B_{r_3}(p_0)$. Given a point $(x,\, y_1) \in B_{r_3}(p_0)$, let $t_1 = t(x, y_1)$. Then $y_1 = y^{t_1}(x)$. Similarly for $(x,\, y_2)$ near $(x,\, y_1)$ and $y_2 > y_1$, let $t_2 = t(x, y_2)$. Then $y_2 = y^{t_2}(x)$ and $t_2 > t_1$. We want to compute

$$\frac{t(x, y_2) - t(x, y_1)}{y_2 - y_1} = \frac{t_2 - t_1}{y_2 - y_1} = \frac{t_2 - t_1}{y^{t_2}(x) - y^{t_1}(x)}$$

Now let

$$B(x) \equiv \int_{y^{t_1}(x)}^{y^{t_2}(x)} D(x, y) \sin\theta(x, y) dy. \tag{4.19}$$

We want to know how $B(x)$ is related to $B(0)$. Without loss of generality we may assume that $x > 0$. Instead of (4.6) we have

$$\oint_{\partial\tilde{\Omega}} DN^{\perp} \cdot \nu = \int_{\tilde{\Omega}} rot\vec{F} \tag{4.20}$$

by letting $\varphi = 1$ in (3.18), where $\tilde{\Omega} \equiv \{\ (\varsigma,\, \eta) \mid 0 < \varsigma < x,\ y^{t_1}(\varsigma) < \eta < y^{t_2}(\varsigma)\}$. Observe that $\nu = \pm N$ along the characteristic curves $\{t = t_j\}$, $j = 1,\, 2$ while $\nu = (1,\, 0)$ on $\{\ (x, y) \mid y^{t_1}(x) < y < y^{t_2}(x)\}$ and $\nu = (-1,\, 0)$ on $\{\ (0, y) \mid t_1 < y < t_2\}$. It follows from (4.20) that (recall that $N^{\perp} = (\sin\theta,\, -\cos\theta)$)

$$\begin{aligned} &\int_{y^{t_1}(x)}^{y^{t_2}(x)} D(x, y) \sin\theta(x, y) dy + \int_{t_2}^{t_1} (-D(0, y) \sin\theta(0, y)) dy \\ = \ &\int_0^x \Big(\int_{y^{t_1}(\varsigma)}^{y^{t_2}(\varsigma)} rot\vec{F}\ d\eta\Big) d\varsigma. \end{aligned} \tag{4.21}$$

Then by (4.21) and (4.19) we deduce that

$$B'(x) = \int_{y^{t_1}(x)}^{y^{t_2}(x)} rot\vec{F}\ (x, y) dy = rot\vec{F}(x, \eta')(y^{t_2}(x) - y^{t_1}(x)). \tag{4.22}$$

and

$$B(x) = (y^{t_2}(x) - y^{t_1}(x)) D(x, \eta) \sin\theta(x, \eta) \tag{4.23}$$

for $\eta' = \eta'(x,\, t_1,\, t_2)$, $\eta = \eta(x,\, t_1,\, t_2)$ such that $y^{t_1}(x) < \eta,\, \eta' < y^{t_2}(x)$ by the mean value theorem. By (4.22) and (4.23) we obtain

$$\frac{d \log B(x)}{dx} = \frac{B'(x)}{B(x)} = \frac{rot\vec{F}(x, \eta')}{D(x, \eta) \sin\theta(x, \eta)} \tag{4.24}$$

(noting that $\sin\theta$ is close to 1 near p_0 where θ equals $\frac{\pi}{2}$ by assumption). Integrating both sides of (4.24) gives

$$\frac{B(x)}{B(0)} = \exp \int_0^x \frac{rot\vec{F}(\varsigma, \eta')}{D(\varsigma, \eta) \sin\theta(\varsigma, \eta)} d\varsigma. \tag{4.25}$$

Observe that $B(0) = (t_2 - t_1)\ D(0,\ \eta(0, t_1, t_2)) \sin\theta(0,\ \eta(0, t_1, t_2))$ by (4.23). It then follows from (4.25) that

$$\begin{aligned} &\frac{t_2 - t_1}{y^{t_2}(x) - y^{t_1}(x)} \\ = \ &\frac{D(x, \eta(x)) \sin\theta(x, \eta(x))}{D(0, \eta(0)) \sin\theta(0, \eta(0))} \exp\Big(-\int_0^x \frac{rot\vec{F}(\varsigma, \eta'(\varsigma))}{D(\varsigma, \eta(\varsigma)) \sin\theta(\varsigma, \eta(\varsigma))} d\varsigma\Big). \end{aligned} \tag{4.26}$$

We have omitted the dependence of t_1 and t_2 for the expression of η and η' in (4.26). Combining (4.4) with s replaced by t and (4.26) and taking the limit $y_2 \to y_1$, we finally obtain

$$\frac{\partial t}{\partial y}(x, y_1) = \frac{D(x, y_1)\sin\theta(x, y_1)}{D(0, t_1)\sin\theta(0, t_1)}\exp(-\int_0^x \frac{rot\vec{F}(\varsigma, y^{t_1}(\varsigma))}{D(\varsigma, y^{t_1}(\varsigma))\sin\theta(\varsigma, y^{t_1}(\varsigma))}d\varsigma). \tag{4.27}$$

Here we have used Lebesgue's Dominated Convergence Theorem since $\frac{rot\vec{F}(\varsigma,\eta'(\varsigma))}{D(\varsigma,\eta(\varsigma))\sin\theta(\varsigma,\eta(\varsigma))}$ is uniformly bounded and both η' and η converge to y^{t_1} pointwise. Let σ denote the arc length (unit-speed) parameter of the characteristic curve $\{t = t_1\}$. By (4.18), the definition of t, we have $\frac{\partial t}{\partial \sigma} = 0$. Note that $N^{\perp}d\sigma = (dx,\ dy)$ along $\{t = t_1\} = \{\ (x,\ y^{t_1}(x))\ \}$. It then follows that

$$\begin{aligned} \frac{\partial t}{\partial x}(x, y_1) &= -\frac{\partial t}{\partial y}(x, y_1)\frac{dy^{t_1}(x)}{dx} = -\frac{\partial t}{\partial y}(x, y_1)(-\frac{\cos\theta(x, y_1)}{\sin\theta(x, y_1)}) \\ &= g(x, y_1)D(x, y_1)\cos\theta(x, y_1) \end{aligned} \tag{4.28}$$

by (4.27), where

$$g(x, y_1) \equiv \frac{1}{D(0, t_1)\sin\theta(0, t_1)}\exp(-\int_0^x \frac{rot\vec{F}(\varsigma, y^{t_1}(\varsigma))}{D(\varsigma, y^{t_1}(\varsigma))\sin\theta(\varsigma, y^{t_1}(\varsigma))}d\varsigma). \tag{4.29}$$

Suppose $\sigma = 0$ at $(0,\ t_1)$ and $\sigma = l$ at (x, y_1). Recall that $N^{\perp}d\sigma = (dx,\ dy)$ and hence $\sin\theta(\varsigma, y^{t_1}(\varsigma))\ d\sigma = d\varsigma$. So we can rewrite (4.29) as

$$g(x, y_1) = g(0, t_1)\exp(-\int_0^l \frac{rot\vec{F}(\Gamma_{(0,t_1)}(\sigma))}{D(\Gamma_{(0,t_1)}(\sigma))}d\sigma) \tag{4.30}$$

where $\Gamma_{(0,t_1)}$ denotes the characteristic curve $\{t = t_1\}$ from $(0, t_1)$ to (x, y_1), parametrized by σ. Since $t \in C^0$, $\frac{rot\vec{F}(\varsigma,y^{t_1}(\varsigma))}{D(\varsigma,y^{t_1}(\varsigma))\sin\theta(\varsigma,y^{t_1}(\varsigma))}$ is uniformly bounded and y^{t_2} converges to y^{t_1} pointwise as $y_2 \to y_1$, we can apply Lebesgue's Dominated Convergence Theorem to (4.29) and conclude that g is continuous in y_1. On the other hand, g is continuous along the characteristic curve in view of (4.30). Together we have $g \in C^0$ near p_0 since the characteristic curves are transverse to the y-axes $\{x = c$, constants$\}$ near p_0. From (4.28), (4.27), and (4.29), we have proved $t \in C^1$ and

$$\nabla t = gD(\cos\theta, \sin\theta) = gDN$$

with $g > 0$ and $g \in C^0$ near p_0. Moreover, recall that $(N^{\perp}g)(x, y_1) \equiv dg(\Gamma_{(0,t_1)}(\sigma))/d\sigma$ at $\sigma = l$ (if exists). Now (4.17) easily follows from (4.30).

□

Proof. **(of Theorem C)** The existence of s, t and (1.9), (1.10) follow from Theorem 4.1 and Theorem 4.2 in view of Theorem 3.2. By (1.9) and (1.10) we learn that the Jacobian of Ψ does not vanish. So by the inverse function theorem Ψ is a C^1 diffeomorphism near p_0 and (1.11) follows from (1.9).

□

Proof. **(of Corollary C.1)** Observe that by (1.6) we have

$$y'(x) = -\frac{\cos\theta}{\sin\theta} \tag{4.31}$$

(Recall that we have assumed that the characteristic curve Γ is a piece of a C^1 smooth graph $(x,\, y(x))$ so that $\sin\theta > 0$ without loss of generality). Since $\sin\theta > 0$, we obtain from (4.31) that

$$\frac{y'(x)}{\sqrt{1+(y'(x))^2}} = -\cos\theta. \tag{4.32}$$

In the proof of Theorem A, we have shown that the left-hand side of (4.32) is C^1 smooth in x. It follows that θ is also C^1 smooth in x and by (2.9) we obtain

$$(\sin\theta)\theta_x = -H. \tag{4.33}$$

Recall that σ denotes the arc-length parameter. Along Γ we have

$$\frac{d\theta}{d\sigma} = \theta_x \frac{dx}{d\sigma} = \theta_x \sin\theta \tag{4.34}$$

in view of (1.6). From (1.11) we have $(\Psi^*)\ ds = f d\sigma$ along Γ which is defined by t = constant. Hence we can compute

$$\theta_s = \frac{1}{f}\frac{d\theta}{d\sigma} = -\frac{H}{f}$$

by (4.34) and (4.33).

□

5. Regularity of θ and characteristic and seed curves

In Section 4, for a C^1 smooth solution u to (1.3) with $\vec{F} \in C^1(\Omega)$, $H \in C^0(\Omega)$, and Ω nonsingular, we can find locally C^1 smooth characteristic coordinates s, t, that is, a local coordinate change $\Psi : (x,\, y) \to (s,\, t)$ which is a C^1 smooth diffeomorphism such that $\{s = \text{constants}\}$ are seed curves and $\{t = \text{constants}\}$ are characteristic curves. In this section we will prove that if $N(H)$ exists and is in C^0, then θ is C^1 smooth with respect to x, y coordinates. This is equivalent to proving that θ is C^1 smooth with respect to s, t coordinates since Ψ is a C^1 smooth diffeomorphism. So we consider only s, t coordinates throughout this section.

Definition 5.1. *Let Ω_1 and Ω be domains of R^2 such that $\Omega_1 \subset\subset \Omega$. We call s, t C^1 coordinates of $\bar{\Omega}_1$ if there exists a domain Ω_2 such that $\Omega_1 \subset\subset \Omega_2 \subset \Omega$ and s, t are C^1 coordinates of Ω_2, i.e., the coordinate change $\Psi : (x,\, y) \to (s,\, t)$ is a C^1 smooth diffeomorphism onto Ω_2.*

Lemma 5.1. *Let Ω_1 and Ω be domains of R^2 such that $\Omega_1 \subset\subset \Omega$. Let s, t be C^1 coordinates of $\bar{\Omega}_1$ and $\Omega_1 = (0,\, \tilde{s}) \times (0,\, \tilde{t})$ for some $\tilde{s}$, $\tilde{t} > 0$. Suppose $h \in C^0(\Omega)$, and for points in $\bar{\Omega}_1$, h is C^1 smooth in s and $h_s \equiv \frac{\partial h}{\partial s}$ is C^1 smooth in t. Let $(s_2,\, t_2)$, $(s_1,\, t_2)$, $(s_2,\, t_1)$, $(s_1,\, t_1) \in \bar{\Omega}_1$ with $t_2 \neq t_1$.*

(a) Let $M > 0$ be a given constant. Let $s_0 = \min\{\tilde{s},\, \frac{M}{2}(\max_{\bar{\Omega}_1} |(h_s)_t| + 1)^{-1}\}$. Then if $\frac{h(s_1,t_2)-h(s_1,t_1)}{t_2-t_1} > M > 0$ (or $< -M$, resp.), then $\frac{h(s_2,t_2)-h(s_2,t_1)}{t_2-t_1} > \frac{M}{2}$ ($< -\frac{M}{2}$, resp.) for $|s_2 - s_1| \leq s_0$.

(b) Given s_0', $0 \le s_0' \le \tilde{s}$. *Let* $M_0 = 2s_0' \max_{\bar{\Omega}_1} |(h_s)_t|$. *Then for any* $M \ge M_0$, $|s_2 - s_1| \le s_0'$, *if* $\frac{h(s_1,t_2)-h(s_1,t_1)}{t_2-t_1} > M$ *(or* $< -M$*, resp.), then* $\frac{h(s_2,t_2)-h(s_2,t_1)}{t_2-t_1} > \frac{M}{2}$ *(*$< -\frac{M}{2}$*, resp.).*

Recall that by h being C^1 smooth in s we mean that $h_s \equiv \frac{\partial h}{\partial s}$ exists and is continuous. Since we only assume that h is C^1 smooth in s, we need to consider the behavior of h with respect to t in order to prove $h \in C^1$ for later applications. Therefore we study the properties of difference quotient $\frac{h(s,t_2)-h(s,t_1)}{t_2-t_1}$.

Proof. We can write

$$h(s_2,t) - h(s_1,t) = \int_{s_1}^{s_2} h_s(s,t)ds \tag{5.1}$$

by the fundamental theorem of calculus. With $t = t_2$ and t_1 in (5.1) respectively, we then substract one resulting formula from the other to get

$$\frac{h(s_2,t_2)-h(s_2,t_1)}{t_2-t_1} - \frac{h(s_1,t_2)-h(s_1,t_1)}{t_2-t_1} = \int_{s_1}^{s_2} \frac{h_s(s,t_2)-h_s(s,t_1)}{t_2-t_1} ds. \tag{5.2}$$

By the mean value theorem we can find $t' = t'(s)$ between t_1 and t_2 such that

$$\frac{h_s(s,t_2)-h_s(s,t_1)}{t_2-t_1} = (h_s)_t(s,t') \tag{5.3}$$

since h_s is C^1 smooth in t by assumption. Substituting (5.3) into (5.2), we obtain

$$\frac{h(s_2,t_2)-h(s_2,t_1)}{t_2-t_1} - \frac{h(s_1,t_2)-h(s_1,t_1)}{t_2-t_1} = \int_{s_1}^{s_2} (h_s)_t(s,t'(s))ds. \tag{5.4}$$

From (5.4) we can easily deduce (a) and (b). □

Lemma 5.2. *Suppose we have the same situation as in Lemma 5.1.*

(a) Given $\varepsilon > 0$ *we can find* $\delta_0 = \delta_0((h_s)_t|_{\bar{\Omega}_1}, \varepsilon) > 0$ *such that for* p, $q \in \bar{\Omega}_1$, $|p - q| < \delta_0$, *there holds*

$$|(h_s)_t(p) - (h_s)_t(q)| < \varepsilon. \tag{5.5}$$

(b) Let (s_2, t_3), (s_1, t_3), (s_2, t_2), (s_1, t_2), (s_2, t_1), $(s_1, t_1) \in \bar{\Omega}_1$ *with* $t_3 \ne t_1$, $t_2 \ne t_1$. *Given* $\varepsilon > 0$. *Then for* $|t_3 - t_1| + |t_2 - t_1| < \delta_0$ *(as in (a)), there holds*

$$\begin{aligned} &|[\frac{h(s_2,t_3)-h(s_2,t_1)}{t_3-t_1} - \frac{h(s_2,t_2)-h(s_2,t_1)}{t_2-t_1}] \\ &-[\frac{h(s_1,t_3)-h(s_1,t_1)}{t_3-t_1} - \frac{h(s_1,t_2)-h(s_1,t_1)}{t_2-t_1}]| \\ \le\; & |s_2 - s_1|\varepsilon. \end{aligned} \tag{5.6}$$

Proof. (a) follows by observing that $(h_s)_t$ is uniformly continuous on $\bar{\Omega}_1$ since it is continuous on $\bar{\Omega}_1$ and $\bar{\Omega}_1$ is compact. For the proof of (b), following the proof of Lemma 5.1, similarly to (5.4), we can find t'' between t_1 and t_3 such that

$$\frac{h(s_2,t_3)-h(s_2,t_1)}{t_3-t_1} - \frac{h(s_1,t_3)-h(s_1,t_1)}{t_3-t_1} = \int_{s_1}^{s_2} (h_s)_t(s,t'')ds. \tag{5.7}$$

Substracting (5.4) from (5.7), we can then estimate

$$\begin{aligned} & |[\frac{h(s_2,t_3)-h(s_2,t_1)}{t_3-t_1} - \frac{h(s_1,t_3)-h(s_1,t_1)}{t_3-t_1}] \\ & -[\frac{h(s_2,t_2)-h(s_2,t_1)}{t_2-t_1} - \frac{h(s_1,t_2)-h(s_1,t_1)}{t_2-t_1}]| \\ \leq & \int_{s_1}^{s_2} |(h_s)_t(s,t'')-(h_s)_t(s,t')|ds \\ \leq & |s_2-s_1|\varepsilon \end{aligned}$$

by (5.5) since $|(s,t'')-(s,t')| = |t''-t'| \leq |t_3-t_1| + |t_2-t_1| \leq \delta_0$.

$\square$

Lemma 5.3. *Let $u : \Omega \subset R^2 \to R$ be a C^1 smooth solution to (1.3) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$ such that Ω is nonsingular. Let $p_0 \in \Omega$ and θ (defined locally) $\in C^0$ such that $(\cos\theta, \sin\theta) = N^u_{\vec{F}}$ with $\theta(p_0) = \frac{\pi}{2}$. Then*

(a) there exists a domain Ω_1 such that $p_0 \in \Omega_1 \subset\subset \Omega$, $\bar{\Omega}_1$ has C^1 coordinates s, t, $\Omega_1 = (0,\tilde{s}) \times (0,\tilde{t})$ in s, t coordinates, and $|\theta - \frac{\pi}{2}| << 1$ in $\bar{\Omega}_1$.

(b) Take s, t coordinates as in (a). Suppose H is C^1 smooth in t. Then there exists a constant $M = M(\tilde{s}, \max_{\bar{\Omega}_1}|y_t|, \max_{\bar{\Omega}_1}|f|, \max_{\bar{\Omega}_1}|\frac{f_t}{f^2}|, \max_{\bar{\Omega}_1}|(\theta_s)_t|) > 0$ such that

$$| \frac{\theta(s,t_2)-\theta(s,t_1)}{t_2-t_1} | \leq M \tag{5.8}$$

for any (s, t_2), $(s, t_1) \in \bar{\Omega}_1$ and $t_2 \neq t_1$.

Proof. (a) follows from Theorem C and $\theta \in C^0$. To prove (b), by (1.11) we first rewrite the equations (1.6) of the characteristic curves in (s, t) coordinates:

$$\begin{aligned} \frac{dx(s,t)}{ds} &= \frac{\sin\theta(s,t)}{f(s,t)}, \\ \frac{dy(s,t)}{ds} &= -\frac{\cos\theta(s,t)}{f(s,t)}. \end{aligned} \tag{5.9}$$

From the second formula of (5.9) we have, for $0 \leq s_1$, $s_2 \leq \tilde{s}$, $0 \leq t_1$, $t_2 \leq \tilde{t}$ and $t_2 \neq t_1$,

$$\begin{aligned} & \frac{y(s_2,t_2)-y(s_2,t_1)}{t_2-t_1} - \frac{y(s_1,t_2)-y(s_1,t_1)}{t_2-t_1} \\ = & \frac{y(s_2,t_2)-y(s_1,t_2)}{t_2-t_1} - \frac{y(s_2,t_1)-y(s_1,t_1)}{t_2-t_1} \\ = & \int_{s_1}^{s_2} \frac{1}{t_2-t_1}[\frac{\cos\theta(s,t_1)}{f(s,t_1)} - \frac{\cos\theta(s,t_2)}{f(s,t_2)}]ds \\ = & \int_{s_1}^{s_2} (A+B)ds. \end{aligned} \tag{5.10}$$

where

$$A = \frac{-1}{f(s,t_2)}\frac{\cos\theta(s,t_2)-\cos\theta(s,t_1)}{t_2-t_1}, \tag{5.11}$$
$$B = -\frac{\cos\theta(s,t_1)}{t_2-t_1}[\frac{1}{f(s,t_2)}-\frac{1}{f(s,t_1)}].$$

Observe that

$$A = \frac{\sin\theta'}{f(s,t_2)}\frac{\theta(s,t_2)-\theta(s,t_1)}{t_2-t_1} \tag{5.12}$$

where θ' is a number between $\theta(s,t_2)$ and $\theta(s,t_1)$ and

$$B = \frac{\cos\theta(s,t_1)}{f^2(s,t')}\frac{\partial f}{\partial t}(s,t') \tag{5.13}$$

where t' is a number between t_2 and t_1 by the mean value theorem (noting that $f > 0$ is C^1 smooth in t by Theorem C). Since $|\theta - \frac{\pi}{2}| << 1$ in $\bar{\Omega}_1$, we have

$$\frac{\sin\theta'}{f(s,t_2)} \geq C_1 \equiv \frac{1}{2\max_{\bar{\Omega}_1}|f|}, \quad |\, B \,| \leq C_2 \equiv \max_{\bar{\Omega}_1}|\frac{f_t}{f^2}| \tag{5.14}$$

in $\bar{\Omega}_1$.

On the other hand, we have

$$\left|\frac{y(s_2,t_2)-y(s_2,t_1)}{t_2-t_1} - \frac{y(s_1,t_2)-y(s_1,t_1)}{t_2-t_1}\right| \tag{5.15}$$
$$\leq 2\max_{\bar{\Omega}_1}|y_t| \leq C_3 \equiv 2\max_{\bar{\Omega}_1}|y_t| + 1$$

since y is C^1 smooth on $\bar{\Omega}_1$. Since $\theta_s = -\frac{H}{f}$ (see Corollary C.1), $(\theta_s)_t = -(\frac{H}{f})_t \in C^0$ by the assumption on H and f being C^1 in t due to Theorem C. So the assumptions of Lemma 5.1 are satisfied for $h = \theta$. Take $s_0' = \tilde{s}$ in Lemma 5.1 (b). Since $0 \leq s_1$, $s_2 \leq \tilde{s}$, we have $|s_2 - s_1| \leq \tilde{s}$. Let $M = \max\{M_0 \equiv 2\tilde{s}\max_{\bar{\Omega}_1}|(\theta_s)_t|, \frac{2(2C_3+\tilde{s}C_2)}{\tilde{s}C_1}\}$. Suppose (5.8) fails to hold. Then we can find (s_1, t_2), $(s_1, t_1) \in \bar{\Omega}_1$, $t_2 \neq t_1$, such that

$$\frac{\theta(s_1,t_2)-\theta(s_1,t_1)}{t_2-t_1} > M \text{ (or } < -M, \text{ resp.).}$$

Applying Lemma 5.1(b) to $h = \theta$ gives

$$\frac{\theta(s_2,t_2)-\theta(s_2,t_1)}{t_2-t_1} > \frac{M}{2} \text{ (or } < -\frac{M}{2}, \text{ resp.).} \tag{5.16}$$

for all $s_2 \in [0, \tilde{s}]$. From (5.10), (5.15), (5.14), and (5.16), we estimate

$$C_3 \geq |\frac{y(s_2',t_2)-y(s_2',t_1)}{t_2-t_1} - \frac{y(s_1,t_2)-y(s_1,t_1)}{t_2-t_1}| \tag{5.17}$$
$$= \left|\int_{s_1}^{s_2'}(A+B)ds\right|$$
$$> |s_2'-s_1|(\frac{M}{2}C_1 - C_2) \geq \frac{\tilde{s}}{2}(\frac{M}{2}C_1 - C_2) \geq C_3$$

for some s_2', $0 \leq s_2' \leq \tilde{s}$, satisfying $|s_2' - s_1| \geq \frac{\tilde{s}}{2}$. We have reached a contradiction. Therefore (5.8) holds.

□

The following lemma should be a basic fact in calculus. For completeness we give a proof.

Lemma 5.4. *Let w be a real C^1 smooth function of $(s,\, t) \in \Omega \subset R^2$. Suppose further that $(w_s)_t$ exists and is continuous. Then $(w_t)_s$ exists and equals $(w_s)_t$ (hence is continuous).*

Proof. Take $(s_0,\, t_0) \in \Omega$. Then there exists $r_0 > 0$ such that $B_{r_0}((s_0,\, t_0)) \subset \Omega$. For $(s_1,\, t_1) \in B_{r_0}((s_0,\, t_0))$ we write

$$\begin{aligned} w(s_1,t_1) - w(s_0,t_1) &= \int_{s_0}^{s_1} w_s(s,t_1)ds \\ w(s_1,t_0) - w(s_0,t_0) &= \int_{s_0}^{s_1} w_s(s,t_0)ds. \end{aligned} \tag{5.18}$$

Take the difference of the two equalities in (5.18) divided by $t_1 - t_0$. Let t_1 approach t_0 in the resulting formula to get

$$w_t(s_1,t_0) - w_t(s_0,t_0) = \int_{s_0}^{s_1} (w_s)_t(s,t_0)ds. \tag{5.19}$$

Now dividing (5.19) by $s_1 - s_0$ and applying the mean value theorem to $(w_s)_t$, we obtain

$$\begin{aligned} \frac{w_t(s_1,t_0) - w_t(s_0,t_0)}{s_1 - s_0} &= \frac{1}{s_1 - s_0}\int_{s_0}^{s_1} (w_s)_t(s,t_0)ds \\ &= (w_s)_t(\tilde{s},t_0) \end{aligned} \tag{5.20}$$

for $s_0 < \tilde{s} < s_1$. Letting s_1 tend to s_0 in (5.20), we obtain the existence of $(w_t)_s(s_0,t_0)$ and

$$(w_t)_s(s_0,t_0) = (w_s)_t(s_0,t_0)$$

since $(w_s)_t$ is continuous.

□

Proof. **(of Theorem D)** Given a point $p_0 \in \Omega$, we may assume $\theta(p_0) = \frac{\pi}{2}$ without loss of generality. By Theorem C and θ (defined locally) $\in C^0$, we can choose a domain Ω_1, $p_0 \in \Omega_1 \subset\subset \Omega$, such that $\bar{\Omega}_1$ has C^1 coordinates s, t, $\Omega_1 = (0,\tilde{s}) \times (0,\tilde{t})$ in s,t coordinates, and $|\theta - \frac{\pi}{2}| << 1$ in $\bar{\Omega}_1$. Since $N = gD\frac{\partial}{\partial t}$ in view of (1.11), we have $H_t = \frac{1}{gD}N(H) \in C^0$. It follows from Lemma 5.3 (b) that

$$\mid \frac{\theta(s,t_2) - \theta(s,t_1)}{t_2 - t_1} \mid \leq M = M(\tilde{s},\ \max_{\bar{\Omega}_1}|y_t|,\ \max_{\bar{\Omega}_1}|f|,\ \max_{\bar{\Omega}_1}|\frac{f_t}{f^2}|,\ \max_{\bar{\Omega}_1}|(\theta_s)_t|)$$

for any $(s,\, t_2)$, $(s,\, t_1) \in \bar{\Omega}_1$ and $t_2 \neq t_1$ (see (5.8)). We claim that for $(s_1,\, t)$, $(s_1,\, t_1) \in \Omega_1$ and $t \neq t_1$,

$$\frac{\theta(s_1,t) - \theta(s_1,t_1)}{t - t_1}$$

is a Cauchy sequence as $t \to t_1$ (if so, $\frac{\partial \theta}{\partial t}$ exists at $(s_1,\ t_1)$). If not, there exists $\kappa > 0$ such that for any positive integer n, there exist t''_n, $t'_n \in (0, \tilde{t})$, $t''_n \neq t_1$, $t'_n \neq t_1$ satisfying $|t''_n - t_1| + |t'_n - t_1| \leq \min(\tilde{t},\ \frac{1}{n})$ and

$$\frac{\theta(s_1,t''_n)-\theta(s_1,t_1)}{t''_n - t_1} - \frac{\theta(s_1,t'_n)-\theta(s_1,t_1)}{t'_n - t_1} > \kappa \text{ (or } < -\kappa, \text{ resp.).} \tag{5.21}$$

Take $h = \theta$ and $\varepsilon = \frac{\kappa}{2\tilde{s}}$ in Lemma 5.2. Then there exists $0 < \delta_0 = \delta_0((\theta_s)_t|_{\bar{\Omega}_1},\ \varepsilon) \leq \tilde{t}$ such that

$$\begin{aligned} & |(\frac{\theta(s_2,t''_n)-\theta(s_2,t_1)}{t''_n - t_1} - \frac{\theta(s_2,t'_n)-\theta(s_2,t_1)}{t'_n - t_1}) \\ & -(\frac{\theta(s_1,t''_n)-\theta(s_1,t_1)}{t''_n - t_1} - \frac{\theta(s_1,t'_n)-\theta(s_1,t_1)}{t'_n - t_1})| \\ \leq \quad & |s_2 - s_1|\varepsilon \leq \frac{\kappa}{2} \end{aligned}$$

for $0 \leq s_2 \leq \tilde{s}$, $|t''_n - t_1| + |t'_n - t_1| \leq \delta_0$ (see (5.6)). So if we require $t''_n \neq t_1$, $t'_n \neq t_1$ satisfying $|t''_n - t_1| + |t'_n - t_1| \leq \min(\tilde{t},\ \frac{1}{n},\ \delta_0)$, we can then estimate

$$\begin{aligned} & \frac{\theta(s_2,t''_n)-\theta(s_2,t_1)}{t''_n - t_1} - \frac{\theta(s_2,t'_n)-\theta(s_2,t_1)}{t'_n - t_1} \\ \geq \quad & \frac{\theta(s_1,t''_n)-\theta(s_1,t_1)}{t''_n - t_1} - \frac{\theta(s_1,t'_n)-\theta(s_1,t_1)}{t'_n - t_1} - \frac{\kappa}{2} \\ > \quad & \kappa - \frac{\kappa}{2} = \frac{\kappa}{2} \ (< -\frac{\kappa}{2}, \text{ resp.)} \end{aligned} \tag{5.22}$$

by (5.21) for $0 \leq s_2 \leq \tilde{s}$. By (5.10) we have

$$\begin{aligned} & \frac{y(s_2,t'_n)-y(s_2,t_1)}{t'_n - t_1} - \frac{y(s_1,t'_n)-y(s_1,t_1)}{t'_n - t_1} \\ = \quad & \int_{s_1}^{s_2} (A'_n + B'_n) ds \end{aligned} \tag{5.23}$$

where

$$\begin{aligned} A'_n &= \frac{-1}{f(s,t'_n)} \frac{\cos\theta(s,t'_n) - \cos\theta(s,t_1)}{t'_n - t_1}, \\ B'_n &= -\frac{\cos\theta(s,t_1)}{t'_n - t_1}[\frac{1}{f(s,t'_n)} - \frac{1}{f(s,t_1)}]. \end{aligned} \tag{5.24}$$

Since $|B'_n| \leq \max_{\bar{\Omega}_1} |\frac{f_t}{f^2}|$ and B'_n converges to $\cos\theta(s,t_1)\ \frac{f_t}{f^2}(s,t_1)$ pointwise, we conclude from Lebesgue's Dominated Convergence Theorem that $\lim_{n\to\infty} \int_{s_1}^{s_2} A'_n ds$ exists and

$$\lim_{n\to\infty} \int_{s_1}^{s_2} A'_n ds = y_t(s_2,t_1) - y_t(s_1,t_1) - \int_{s_1}^{s_2} \cos\theta(s,t_1)\frac{f_t}{f^2}(s,t_1) ds. \tag{5.25}$$

Similarly with t'_n replaced by t''_n in (5.23) and the same reasoning, we also obtain

$$\lim_{n\to\infty} \int_{s_1}^{s_2} A''_n ds = y_t(s_2,t_1) - y_t(s_1,t_1) - \int_{s_1}^{s_2} \cos\theta(s,t_1)\frac{f_t}{f^2}(s,t_1) ds \tag{5.26}$$

where

$$A''_n = \frac{-1}{f(s,t''_n)} \frac{\cos\theta(s,t''_n) - \cos\theta(s,t_1)}{t''_n - t_1}.$$

Therefore by (5.25) and (5.26) we have

$$\lim_{n\to\infty}\int_{s_1}^{s_2}(A_n''-A_n')ds=0. \tag{5.27}$$

On the other hand, using the mean value theorem, we can find $\tilde{\theta}'$ ($\tilde{\theta}''$, resp.) between $\theta(s,t_n')$ ($\theta(s,t_n'')$, resp.) and $\theta(s,t_1)$ such that

$$\begin{aligned} A_n' &= \frac{\sin\tilde{\theta}'}{f(s,t_n')}\frac{\theta(s,t_n')-\theta(s,t_1)}{t_n'-t_1}, \\ A_n'' &= \frac{\sin\tilde{\theta}''}{f(s,t_n'')}\frac{\theta(s,t_n'')-\theta(s,t_1)}{t_n''-t_1}. \end{aligned}$$

Hence we can write

$$\begin{aligned} A_n''-A_n' &= \frac{\sin\tilde{\theta}''}{f(s,t_n'')}[\frac{\theta(s,t_n'')-\theta(s,t_1)}{t_n''-t_1}-\frac{\theta(s,t_n')-\theta(s,t_1)}{t_n'-t_1}] \\ &\quad +(\frac{\sin\tilde{\theta}''}{f(s,t_n'')}-\frac{\sin\tilde{\theta}'}{f(s,t_n')})\frac{\theta(s,t_n')-\theta(s,t_1)}{t_n'-t_1} \end{aligned} \tag{5.28}$$

The second term in the right-hand side of (5.28) is uniformly bounded by Lemma 5.3 (b) and converges to zero pointwise. Therefore the integral (from s_1 to s_2) of this term goes to zero as $n\to\infty$ (while t_n'', $t_n'\to t_1$ and $\tilde{\theta}''$, $\tilde{\theta}'\to\theta(s,t_1)$) by Lebesgue's Dominated Convergence Theorem. But the first term in the right-hand side of (5.28) $\geq\frac{1}{2}(\max_{\bar{\Omega}_1}|f|)^{-1}\frac{\kappa}{2}$ (or $\leq-\frac{1}{2}(\max_{\bar{\Omega}_1}|f|)^{-1}\frac{\kappa}{2}$, resp.) by (5.22). Altogether from (5.28) we have the following estimate

$$|\lim_{n\to\infty}\int_{s_1}^{s_2}(A_n''-A_n')ds|\geq|s_2-s_1|\frac{\kappa}{4}(\max_{\bar{\Omega}_1}|f|)^{-1}$$

which contradicts (5.27). Therefore the claim holds, which implies the existence of $\theta_t\equiv\frac{\partial\theta}{\partial t}$ at any point p_0.

For continuity of θ_t, we start with (5.2) taking $h=\theta$. For $0\leq s_1$, $s'\leq\tilde{s}$, $0\leq t_1$, $t'\leq\tilde{t}$, $t'\neq t_1$, we have

$$\frac{\theta(s',t')-\theta(s',t_1)}{t'-t_1}-\frac{\theta(s_1,t')-\theta(s_1,t_1)}{t'-t_1}=\int_{s_1}^{s'}\frac{\theta_s(s,t')-\theta_s(s,t_1)}{t'-t_1}ds \tag{5.29}$$

where $\theta_s=-\frac{H}{f}\in C^1$ in t by (1.12), Theorem C, and the assumption (note that $NH=gD\frac{\partial}{\partial t}H$). Taking the limit $t'\to t_1$ in (5.29), we obtain

$$\theta_t(s',t_1)-\theta_t(s_1,t_1)=\int_{s_1}^{s'}(\theta_s)_t(s,t_1)ds. \tag{5.30}$$

It follows from (5.30) that θ_t is continuous in the s direction at (s_1,t_1), for all $0\leq s_1\leq\tilde{s}$, $0\leq t_1\leq\tilde{t}$. So to prove θ_t is continuous at $(s_1,\,t_1)$, it suffices to show that θ_t is continuous in the t direction at $(s_1,\,t_1)$, for all $0\leq s_1\leq\tilde{s}$, $0\leq t_1\leq\tilde{t}$. Suppose this fails to hold. Then there exists $M'>0$ such that for any positive integer m, we can find $t_m'\neq t_1$ satisfying $|t_m'-t_1|<\min\{\tilde{t},\,\frac{1}{m}\}$ and

$$\theta_t(s_1,t_m')-\theta_t(s_1,t_1)>M' \text{ (or } <-M',\text{ resp.).} \tag{5.31}$$

We take the difference of the formula (5.30) with t_1 replaced by t'_m and the formula (5.30) itself to obtain

$$\theta_t(s',t'_m) - \theta_t(s',t_1) = \theta_t(s_1,t'_m) - \theta_t(s_1,t_1) + \int_{s_1}^{s'} ((\theta_s)_t(s,t'_m) - (\theta_s)_t(s,t_1))ds. \tag{5.32}$$

For $|s'-s_1| \leq \min\{\frac{M'}{4}(\max_{\bar{\Omega}_1}|(\theta_s)_t|)^{-1}, \tilde{s}\}$ we then have

$$\theta_t(s',t'_m) - \theta_t(s',t_1) > \frac{M'}{2} \text{ (or } < -\frac{M'}{2}, \text{ resp.)} \tag{5.33}$$

by (5.31) and (5.32). On the other hand, we have a similar formula as (5.25):

$$\lim_{m\to\infty}\int_{s_1}^{s'} A'_m ds = y_t(s',t_1) - y_t(s_1,t_1) - \int_{s_1}^{s'} \cos\theta(s,t_1)\frac{f_t}{f^2}(s,t_1)ds.$$

where

$$A'_m = \frac{-1}{f(s,t'_m)}\frac{\cos\theta(s,t'_m) - \cos\theta(s,t_1)}{t'_m - t_1}.$$

Since A'_m is uniformly bounded (θ is Lipschitzian) and converges pointwise as $m \to \infty$ (θ_t exists), we obtain

$$y_t(s',t_1) - y_t(s_1,t_1) = \int_{s_1}^{s'} \frac{\sin\theta(s,t_1)}{f(s,t_1)}\theta_t(s,t_1) + \int_{s_1}^{s'} \cos\theta(s,t_1)\frac{f_t}{f^2}(s,t_1)ds \tag{5.34}$$

by Lebesgue's dominated convergence theorem. Replacing t_1 by t'_m in (5.34), taking the difference of the resulting formula and (5.34), and letting $m \to \infty$, we finally obtain

$$\lim_{m\to\infty}\int_{s_1}^{s'} \frac{\sin\theta(s,t_1)}{f(s,t_1)}(\theta_t(s,t'_m) - \theta_t(s,t_1))ds = 0 \tag{5.35}$$

by a similar reasoning as before. On the other hand, we estimate

$$\begin{aligned} &\frac{\sin\theta(s,t_1)}{f(s,t_1)}(\theta_t(s,t'_m) - \theta_t(s,t_1)) \\ \geq\ &\frac{1}{2\max_{\bar{\Omega}_1}|f|}\frac{M'}{2} \text{ (or } \leq \frac{-1}{2\max_{\bar{\Omega}_1}|f|}\frac{M'}{2}, \text{ resp.)} \end{aligned}$$

by (5.33), which contradicts (5.35). Thus we have shown that θ_t is continuous. Since θ_s is continuous by Corollary C.1, we conclude that $\theta \in C^1$ in s, t coordinates, and hence $\theta \in C^1$ in x, y coordinates. It follows that the characteristic curves are then C^2 smooth in view of (1.6). Similarly the seed curves are also C^2 smooth.

To compute θ_t, we first show that D_s $(= f^{-1}N^{\perp}(D))$ exists and is continuous. Observe that

$$x_s = \frac{\sin\theta}{f}, x_t = \frac{\cos\theta}{gD}, y_s = -\frac{\cos\theta}{f}, y_t = \frac{\sin\theta}{gD} \tag{5.36}$$

and $(x_s)_t$ ($(y_s)_t$,resp.) exists and is continuous since $\theta \in C^1$ and f is C^1 in t by Theorem C. It follows from Lemma 5.4 that $(x_t)_s$ ($(y_t)_s$, resp.) exists and equals

$(x_s)_t$ ($(y_s)_t$, resp.). So by (5.36) D_s exists and is continuous (either $\cos\theta \neq 0$ or $\sin\theta \neq 0$). Now computing $x_{st} = x_{ts}$ and $y_{st} = y_{ts}$, we get

$$\frac{(\cos\theta)\theta_t f - f_t \sin\theta}{f^2} = \frac{-(\sin\theta)\theta_s gD - (gD)_s \cos\theta}{(gD)^2} \tag{5.37}$$

and

$$\frac{(\sin\theta)\theta_t f - f_t(-\cos\theta)}{f^2} = \frac{(\cos\theta)\theta_s gD - (gD)_s \sin\theta}{(gD)^2}. \tag{5.38}$$

Multiply (5.37) by $\cos\theta$ and (5.38) by $\sin\theta$, respectively, and then add the resulting equalities to obtain

$$\frac{\theta_t f}{f^2} = \frac{-(gD)_s}{(gD)^2}. \tag{5.39}$$

We can now compute

$$\begin{aligned}
\theta_t &= -\frac{f g_s}{g^2 D} - \frac{f D_s}{g D^2} \\
&= \frac{(rot\vec{F})g}{(gD)^2} - \frac{f D_s}{g D^2} \text{ (by (1.10) and noting that } N^{\perp} g = f g_s) \\
&= \frac{rot\vec{F}}{gD^2} - \frac{N^{\perp}(\log D)}{gD}
\end{aligned}$$

which is (1.13).

□

We remark that seed curves may only be C^1 smooth, but not C^2 smooth if u is only Lipschitzian, but not C^1 smooth even in the case of $H = 0$. An example is given by $u(x, y) = xy$ for $y > 0$ and $u = 0$ for $y \leq 0$ with $\vec{F} = (-y, x)$ (see Example 7.2 in [7]). The seed curve Λ_a passing through a point $(a, 0)$, $a \neq 0$, is the union of the straight line $\{x = \pm a\}$ for $y \geq 0$ and the semi-circle of center $(0, 0)$ and radius $|a|$ for $y < 0$. It is easy to see that Λ_a is C^∞ smooth except at $(\pm a, 0)$ where Λ_a is only C^1 smooth, but not C^2 smooth. Note that u is not C^1 smooth at the x-axis while it is a Lipschitzian p-minimizer on any bounded plane domain. For more examples, see ([19]).

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(Jih-Hsin Cheng and Jenn-Fang Hwang) Institute of Mathematics, Academia Sinica, Taipei, Taiwan, R.O.C.

E-mail address, Jih-Hsin Cheng: `cheng@math.sinica.edu.tw`

E-mail address, Jenn-Fang Hwang: `majfh@math.sinica.edu.tw`

(Paul Yang) Department of Mathematics, Princeton University, Princeton, NJ 08544, U.S.A.

E-mail address: `yang@Math.Princeton.EDU`